The Completeness of The Global Shape from Shading Algorithm

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Abstract

A global shape from shading algorithm which develops a technique to merge local shape from shading results obtained around singular points into the complete shape using the mountaineers theorem was recently presented in [2]. In this comment, we enhance this result by proving the completeness and uniqueness obtained by the global shape form shading algorithm.

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In [2], the following global shape from shading algorithm was presented. A local shape from shading algorithm is used to recover the shape around each singular point. The algorithm inspects the behavior of iso-height contours around each singular point. The contours are monitored from the singular point "outwards" until another singular point is encountered. An underlying assumption is that the shape to be recovered is a Morse Function. For such functions, according to the mountaineers theorem [1, 3], the number of extrema located within a closed equal height contour of a smooth surface exceeds by one the number of saddle points within that contour. Therefore, when tracking iso-height contours that start as a small circle around an extremum, the first singular point that the extending contours

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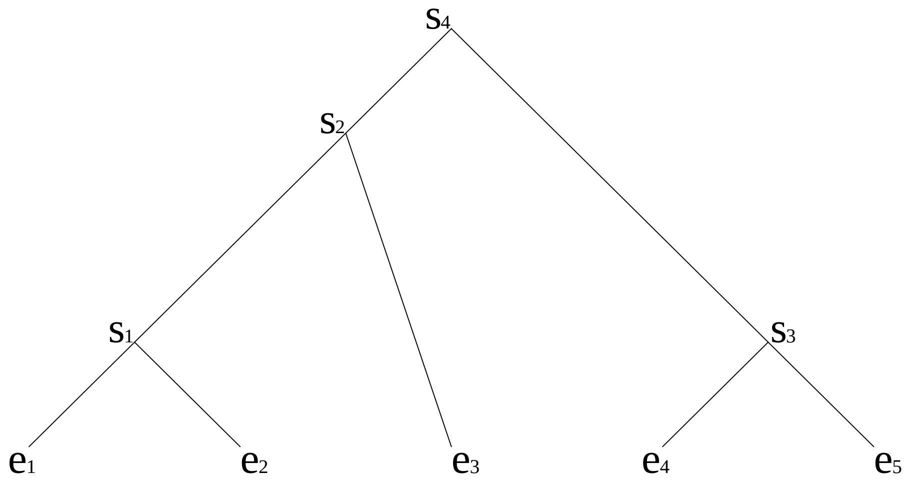

Figure 1: The global algorithm can be described as a binary tree where the leaves are the extrema and the inner nodes are the saddle points closest to the subsurfaces represented by their children.

meet must be a saddle point. This saddle point is the singular point whose height is closest to the height of the extremum. When the illumination is in the direction of the camera, the local shape recovered assuming the extremum is a minimum is the reflection of the shape recovered assuming the extremum is a maximum, therefore the same saddle point will be found. This is not the case when the illumination comes from other directions and the closest saddle point may be different. In both cases the algorithm is run on all possible assignments of extrema as minima or maxima.

The global algorithm merges the results of the local algorithm by merging two local surfaces which have the same closest saddle point. After two surfaces have been merged, the local algorithm extends the merged surface until another saddle point is met. This process continues until all local surfaces have been merged together and the global shape has been recovered.

We assume that on the image boundary the downward direction is everywhere outward (or inward). Under this condition we will prove that only when the singular points are correctly classified will the algorithm be able to complete the recovery of the surface, and that when the singular points are correctly classified there exists a unique solution (modulo the fact that minima and maxima can not be differentiated). Thus the algorithm is run on different classifications of the singular points and when it is able to merge all the subsurfaces, the correct surface has been found.

The first step of the proof will show that when the singular points are correctly classified, the algorithm recovers a unique shape. In the second step of the proof we will show that when the singular points are misclassified the algorithm can not find a way to merge all the local surfaces.

The results of the global algorithm can be described as a binary tree where the leaves are the extrema in the image and the inner nodes are the saddle points closest to the subsurfaces represented by their children. An example of such a tree is given in Figure 1. There is a unique such tree for each image. Although the algorithm can merge the subsurfaces in different orders (e.g. merge the regions of extrema e_1 and e_2 before e_4 and e_5 or vice versa), the resulting tree, and the shape recovered, will always be the same because the closest saddle point to an extremum or subsurface is unique by construction. We will now show that the algorithm will recover a single surface when the singular points are correctly classified.

We will denote by E the set of extrema and S the set of saddle points, and define $n = |S|$. Therefore, according to the mountaineers theorem $|E| = n+1$. For each extremum the saddle point closest to it is detected. As $|E| > |S|$, there must be at least one saddle point s, that has two extrema which s is closest to. Therefore, merging the regions of these two extrema is a step in recovering the correct surface. After these two regions are merged, the number of remaining saddle points is $n-1$, and the number of subregions is n. This argument can be repeated n times, thereby, the algorithm terminates with one surface.

Until now we have assumed that the algorithm correctly classifies the singular points as either extrema or saddle points. We will show now that if some points are misclassified, the algorithm will have to backtrack and will successfully terminate only when the correct classification is made. Using proof by contradiction, let us assume that the algorithm terminates successfully, so that the singular points are classified into E' and S', and that $E' \neq E$ and $S' \neq S$. By the mountaineers theorem, $|E'| = |E| = n+1$ and $|S'| = |S| = n$. Because $|E'| - |S| = 1$, there is at least one extremum point e' such that $e' \in (E \cap E')$. When the local algorithm is run on e', the closest saddle point s' that is found is such that $s' \in (S \cap S')$. From this we conclude that $|E' \cap S| \leq n-1$. Thus $|E' \cap E| \geq 2$. Denote by e'' the new extremum point in $|E' \cap E|$. As before we find the closest saddle point to e'' which we call s''. If it has not already been determined that $s'' \in S \cap S'$, we repeat the argument we made using s'. However if s'' has already been "found" then a whole sub-tree which includes this point is in both solutions has been determined. Therefore, by applying the local algorithm on the subsurface represented by that subtree we find a different saddle point which is closest to that subsurface and apply the argument to it. By repeating these arguments $n+1$ times we show that $E' = E$ and $S' = S$ which is a contradiction. Thus, we have shown that the algorithm will be able to produce a single surface only when the singular points are correctly classified, and as was shown in the previous section, when the singular points are correctly classified a unique solution is obtained.

To illustrate the proof we shall demonstrate it on the example shown in Figure 1. Assume that the first extremum point found in E' is e_1. It follows that $s_1 \in S'$. Assume that the next extremum point found is e_4. It then follows that $s_3 \in S'$. Assume the next extremum point found is e_2. Since $s_1 \in S'$, we know that the whole subtree under s_1 is correctly classified. Therefore by applying the local algorithm to that subtree, we obtain that $s_2 \in S'$. The algorithm continues until it is shown that all the singular points were correctly classified.

The global algorithm is run for each assignment of the extrema as minima or maxima. For each assignment at most one solution will be found. When the lighting direction is parallel to the viewing direction, solutions will be found for the correct assignment and for the inverse assignment which produces the reflection of the correct surface. For other viewing directions, one solution will be found for the correct assignment. For other assignments in both cases a solution might also be found.

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