Recognizing 3D Objects Using Tactile Sensing and Curve Invariants

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Abstract

A general paradigm for recognizing 3D objects is offered, and applied to some geometric primitives (spheres, cylinders, cones, and tori). The assumption is that a curve on the surface, or a pair of intersecting curves, was measured with high accuracy (for instance, by a sensory robot). Differential invariants of the curve(s) are then used to recognize the surface. The motivation is twofold: the output of some devices is not surface range data, but such curves. Also, a considerable speedup is obtained by using curve data, as opposed to surface data which usually contains a much higher number of points.

We survey global, algebraic methods for recognizing surfaces, and point out their limitations. After introducing some notions from differential geometry and elimination theory, the differential and "semi-differential" approach to the problem is described, and novel invariants which are based on the curve's curvature and torsion are derived.

1 Introduction and Previous Work

One task an intelligent system should be able to accomplish is *recognition*. Usually, a recognition system derives some characteristics of an object it examines, and tries to match them against similar characteristics in a data base. Suppose, for instance, that one is dealing with 2D objects, and tries to recognize them, given their boundary. Typically, there is a finite data base these boundaries are matched against; various *invariants* have been derived, some global and some local [1, 17, 26], to solve this problem. These are quantities that do not change under certain transformations (Euclidean, affine, projective), and therefore can be used to recognize an object even after it had been altered by such transformations.

Here, a different problem is addressed – recognizing a surface in 3D space, while the information we have is one-dimensional. Specifically, we assume that some measuring device has sampled a curve, or a pair of intersecting curves, on the surface. Given the curve(s), the goal is to recognize the surface. Typical sensors which are the source of such curves are measuring devices, such as coordinate measuring machines, manufactured by the Brown & Sharpe Company (Figure 1), or the IBM RS/1 Cartesian robot. Such devices can measure 3D curves with very high accuracy (for instance, typical error range for a coordinate measuring machine is 0.01 mm).

In [3], an algorithm is presented for determining the axis of a surface of revolution, using the information measured by a tactile sensor which can also estimate the two principle curvatures (see Section 3.2). Here, we assume that only the data points are given. In [7], the parameters of a cylinder are computed from structured light patterns.

Some previous work has addressed the problem of recognizing various surfaces given their occluding contours [14, 10]. However, the aggregate of possible curves on a surface is, usually, much larger than the aggregate of its occluding contours, and may contain far more complicated curves; for instance, the occluding contour of a sphere is always a circle, while there are a great many 3D curves – some of which have rather complicated structure – on a sphere.

Clearly, we are facing a different type of recognition problem from the one previously described, which is usually solved by matching against a data base. It is impossible to build a data base which contains, say, all the curves on a sphere, or even a dense sampling of these curves. Therefore, we have to discover curve characteristics which will enable to answer a question such as "can this curve, after a certain transformation, be embedded on a sphere?", as opposed to "can this curve, after a certain transformation, be superimposed on curve No.

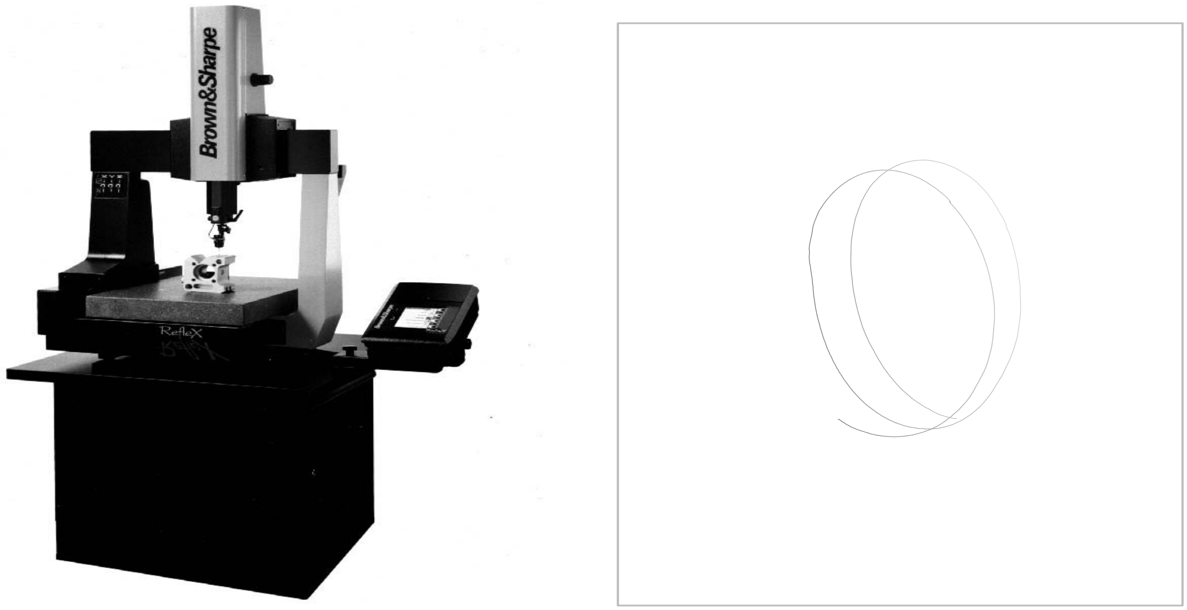


Figure 1: High-accuracy measuring device and a curve it measured on a cylinder.

129 in the data base?".

One way to proceed is straightforward: fit an implicit polynomial to the curve's points, and, from its type, determine the surface. This is the *algebraic approach* [19, 12]. However, this approach will fail if the curve does not lie on a single "primitive" (sphere, cylinder etc), but "crosses over" between two or more primitives (see Section 2.1 and Figure 3). In that case, the global algebraic fit will give us a meaningless result. A very rich theory of *local*, or *differential* invariants, was developed to solve this problem [5, 4, 8, 24, 21]. In Section 2 we quickly survey the global approach as applied to our problems, but the focus of this paper is on the local approach.

Natural curve characteristics to use for recognition are curvature and torsion, as they do not change under rigid transformations. Since we're dealing with 3D data, a rigid transformation is usually a general enough model. So, the goal is to discover invariants depending on a curve's curvature and torsion, which will provide a necessary condition for it to lie on a certain type of surface.

Let us demonstrate this by a simple, 2D example: a plane curve can be embedded in a circle if and only if its curvature is constant. So, in this case, the invariant is the curvature's derivative. Naturally, we don't expect to find such simple invariants for curves lying on 3D surfaces; one trivial example is the well-known condition for a 3D curve to be planar – that its torsion equals zero – but this is an exceptional case.

In the sequel, we derive invariants which are a necessary condition for a curve, or an intersection of two curves, to lie on a sphere, cylinder, cone, or torus. These depend only on the curvature and torsion at a point on the curve (or the curvature and torsion of two curves at their intersection point). We also derive some "semi-differential" invariants, which use not only the differential properties of the curve, but a few points on it. Such invariants have been widely used in computer vision for recognizing plane and space curves [4, 21, 18]; their main advantage is that they allow to use derivatives of lower order than the "purely differential" invariants necessitate.

2 The Algebraic Approach

Implicit polynomials can be used to describe 2D and 3D objects. Some works which address the fitting of implicit polynomials are [22, 2, 13, 23, 20]. One can then use polynomial invariants to recognize the objects [19, 12, 9, 11]. Let us shortly describe how a sphere, cone, cylinder and torus can be recognized using such invariants. Note that the first three objects can be fitted with a quadratic, and the torus with a quartic. Suppose, then, that we succeeded to fit data with a quadratic. Write it as

$$XAX^t + (v, X) + s = 0 \tag{1}$$

where A is a 3×3 matrix, v a vector in $\mathcal{R}^3$, and s a scalar. It is easy to verify that

- If the object is a sphere, A has three positive and identical eigenvalues. It is then trivial to extract the sphere's center and radius.

- If the object is a cylinder, A has two positive and identical eigenvalues, and one zero eigenvalue; also, the axis of the cylinder is in the direction of the eigenvector with zero eigenvalue, and it is trivial to extract its radius.

- If the object is a cone, A has two identical positive eigenvalues and one negative eigenvalue. The axis of the cone is in the direction of the eigenvector with the negative eigenvalue. It is then trivial to extract the cone's opening angle and apex.

- If the object is a torus, its general equation is

$$E_{tor} = ((x-a)^2 + (y-b)^2 + (z-c)^2 + R^2 - r^2)^2 - 4\,R^2((x-a)^2 +$$

$$(y-b)^2+(z-c)^2-((x-a)n_1+(y-b)n_2+(z-c)n_3)^2)$$

where (a,b,c) is its center point, (n_1,n_2,n_3) a unit vector perpendicular to the plane over which the torus lies, and R (r) are the major (minor) radii.

It's trivial to extract a,b,c from E_{tor} (for instance, differentiating E_{tor} three times by x gives $24x-24a$). To extract r and R, note that substituting $\{x=a, y=b, z=c\}$ in E_{tor} gives $r^4+R^4-2R^2r^2$, and substituting $\{x=a, y=b, z=c\}$ in $\frac{\partial^2 E_{tor}}{\partial x^2}+\frac{\partial^2 E_{tor}}{\partial y^2}+\frac{\partial^2 E_{tor}}{\partial z^2}$ gives $-12R^2-12r^2+8R^2{n_1}^2+8R^2{n_2}^2+8R^2{n_3}^2=-4R^2-12r^2$. It is trivial to extract R and r from these two identities. After R, r, a, b, c have been recovered, it is trivial to recover (n_1,n_2,n_3).

2.1 Number of Points Needed

Experiments on curve data show that a relatively high number of points is necessary to achieve reliable algebraic fitting. For instance, for the cylinder data we have used (Figure 1), more than 200 points are required for a reliable fit. We are not sure why this happens; apparently, the fact that the points lie on a curve, which is a "one dimensional entity", results in singularities when trying to fit it with an implicit polynomial which, by its nature, is appropriate for fitting "two dimensional entities".

On the other hand, when using the differential invariants proposed here, a far smaller number of points was necessary; usually, invariants were computed using 10 points or so.

2.2 Applying Invariants to Segmentation

Since the algebraic approach for recognition given a curve may fail, because it can pass through a few geometric primitives, one may try to segment the curve, using some notion of discontinuity, and then use algebraic techniques for each segment. We now show that this is not always easy, by constructing a curve which is infinitely differentiable, yet crosses over from a sphere to a cylinder. Define

$$s(t)=\begin{cases} 0 & t\le 0 \\ \exp(\frac{-1}{t^2}) & t>0 \end{cases}$$

it is well-known that this function is smooth (infinitely differentiable) at every point, and that all its derivatives at $t=0$ are zero. Using $s(t)$, it is trivial to construct smooth functions $s_1(t), s_2(t)$ on the interval $[0,\infty)$ such that $s_1(0)=0, s_1(t)=1$ for $t\ge 1$, $s_2(t)=\sqrt{3}$ for $0\le t\le 1$, and $s_2()$ is monotonically increasing for $t>1$ (see Figure 2).

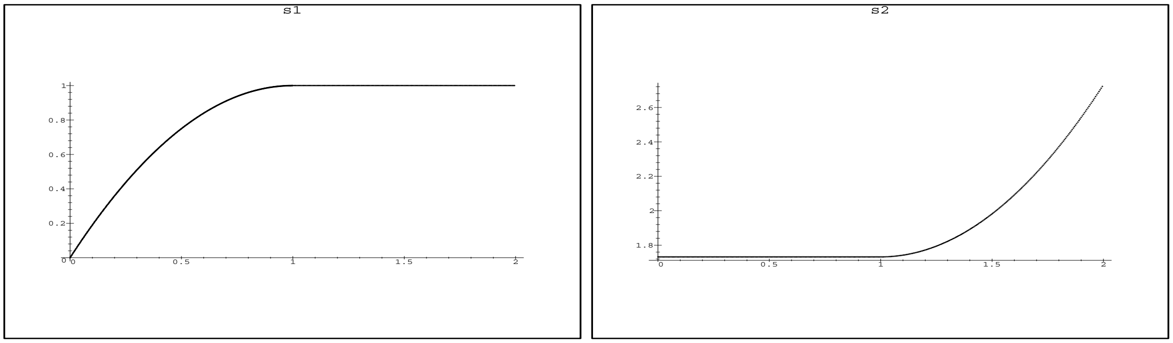


Figure 2: Auxiliary functions used to construct the curve in Figure 3.

Define a curve $c(t)$ as follows

$$c(t) = \begin{cases} (s_1(t)\cos(t), s_1(t)\sin(t), \sqrt{4 - s_1^2(t)}\,) & 0 \le t \le 1 \\ (\cos(t), \sin(t), s_2(t)) & 1 \le t \le 2 \end{cases} .$$

It is easy to see that $c(t)$ is a smooth curve which crosses over from a sphere with radius 2 to a cylinder with radius 1 (at $t = 1$). The curve is displayed in Figure 3. Next to it, we plot the curvature, torsion, curvature's derivative, and a spherical invariant for curves (see Section 5, Equation 10). It is interesting to see that, although the curvature and torsion are continuous, there is a very sharp break in the spherical invariant, at the point in which the curve crosses over from the sphere to the cylinder; this demonstrates that the kind of invariants presented here can succeed where segmentation by "ordinary" differential properties (curvature, torsion etc.) fails.

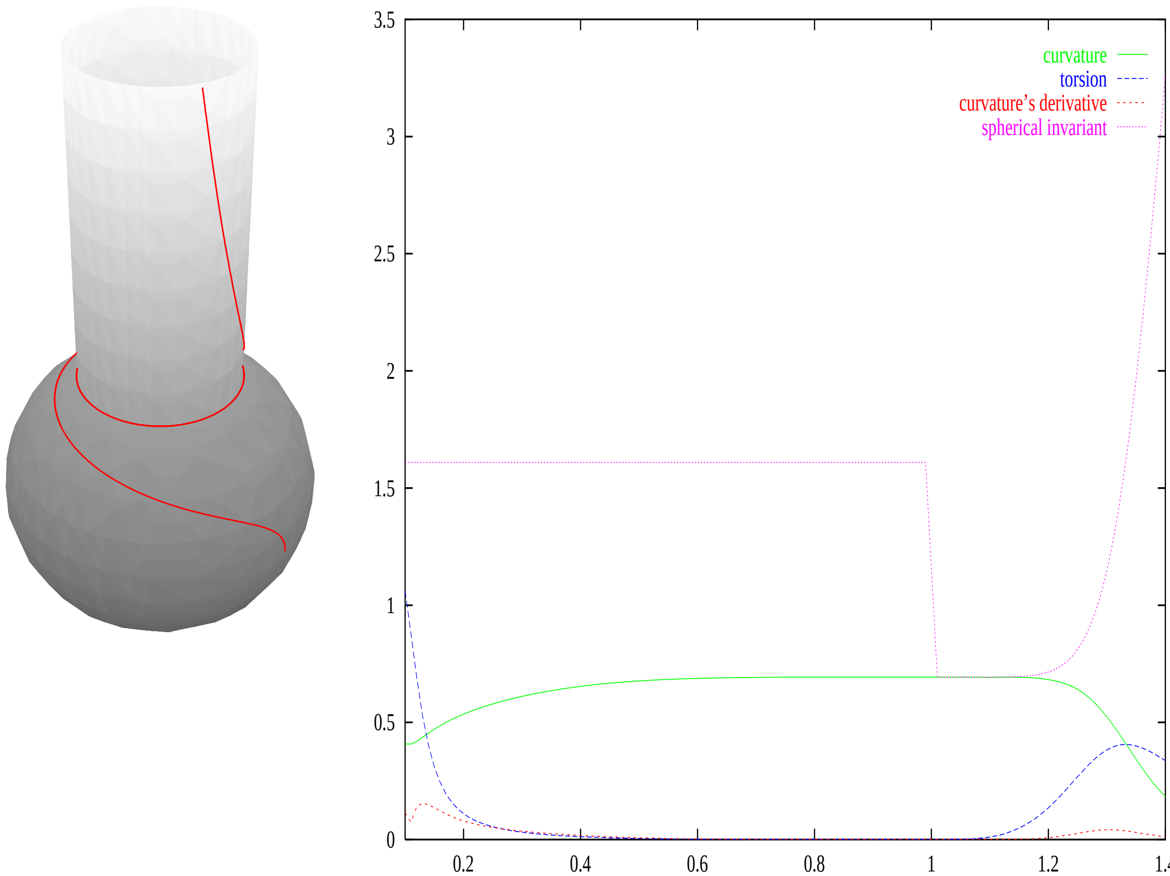


Figure 3: Demonstrating how invariants manage to detect when a curve crosses over from one geometric primitive (sphere) to another (cylinder), although the curvature, torsion etc. cannot detect this crossing over.

3 Mathematical Preliminaries – Some Differential Geometry and Elimination Theory

In the sequel, a few concepts from geometry and algebra are required. We proceed to define them and state some of their important properties.

3.1 Some Differential Geometry of Curves

A curve in 3D Euclidean space is a differentiable function $c : [0,1] \rightarrow \mathcal{R}^3$. At each point $c(t)$, three orthogonal unit vectors are associated with the curve: its *tangent vector* T, which points at the direction of the curve's derivative, its *normal vector* N, and its *binormal vector* B, which is equal to the vector (cross) product of T and N.

This triplet of vectors is called the *Frenet trihedron* at $c(t)$.

In addition, two scalars are associated with each point on the curve. These are the *curvature* κ and *torsion* τ. Intuitively speaking, the curvature measures how "bent" the curve is; for instance, the curvature of a circle is equal to the inverse of its radius. The torsion measures the speed at which the curve moves out of the plane (the so-called *osculating plane*) which locally approximates it; thus, the torsion of a planar curve is zero.

The curvature and torsion can be computed from the parameterization of the curve:

$$\kappa = \frac{|c^{'} \times c^{''}|}{|c^{'}|^3}$$

$$\tau = -\frac{(c^{'} \times c^{''}) \cdot c^{'''}}{|c^{'} \times c^{''}|^2}$$

κ and τ are invariant to translation and rotation; this makes them especially attractive for recognition purposes.

The celebrated *Frenet formulas* relate the Frenet trihedron with the curvature and torsion. If the curve is parameterized by arclength (that is, $|c^{'}| = 1$), the following hold:

$$T^{'} = \kappa N$$

$$N^{'} = -\kappa T - \tau B$$

$$B^{'} = \tau N$$

A concept of crucial importance to this work is the *local canonical form.* Let us see how it is derived. Assume that the curve is parameterized by its arclength s. From Taylor's expansion, we have

$$c(s) = c(0) + sc^{'}(0) + \frac{s^2}{2!}c^{''}(0) + \frac{s^3}{3!}c^{'''}(0) + \frac{s^4}{4!}c^{(4)}(0) + \frac{s^5}{5!}c^{(5)}(0) + o(s^5)$$

$c^{'}(0)$ is equal to the tangent vector T at $t = 0$. Using the first Frenet formula, $c^{''}(0) = T' = \kappa N$. Therefore, $c^{'''}(0) = (\kappa N)' = \kappa^{'} N + \kappa N' = \kappa^{'} N + \kappa(-\kappa T - \tau B) = \kappa^{'} N - \kappa^2 T - \kappa\tau B$.

Similarly, we can derive expressions for the fourth and fifth derivatives. Substituting them into the Taylor series gives

$$\begin{aligned} & c(s) = c(0) + s\,T + \frac{s^2 \kappa N}{2} + \frac{s^3(\kappa^{'}\,N - \kappa^2\,T - \kappa\tau\,B)}{6} + \\ & \frac{s^4(\kappa^{''}\,N - 2\,\kappa^{'}\,\tau\,B - 3\,\kappa\kappa^{'}\,T - \kappa^3 N - \kappa\tau^{'}\,B - \kappa\tau^2 N)}{24} + \\ & \frac{s^5}{120}(\kappa^{'''}\,N - 4\,\kappa\kappa^{''}\,T - 3\,\kappa^{''}\,\tau\,B - 3\,\kappa^{'}\,\tau^{'}\,B - 3\,\kappa^{'}\,\tau^2 N - 3\,\kappa^{'2}\,T - \\ & 6\,\kappa^2\kappa^{'}\,N + \kappa^4\,T + \kappa^3\tau\,B - \kappa\tau^{''}\,B - 3\,\kappa\tau\,\tau^{'}\,N + \kappa^2\tau^2\,T + \kappa\tau^3 B) + o(s^5) \end{aligned} \tag{2}$$

from now on, we shall omit the $o(s^5)$ part. We are allowed to do so as long as the powers of s used are bounded by 5.

3.2 Some Differential Geometry of Surfaces

Locally, a surface S in 3D Euclidean space is a differentiable image of an open set $\mathcal{O}$ in $\mathcal{R}^2$. Formally, it is the set of triplets $\{(x(u,v), y(u,v), z(u,v)) \;/\; (u,v) \in \mathcal{O}\}$. The *tangent plane* to S at the point $((x(u,v), y(u,v), z(u,v))$ is the plane spanned by (x_u, y_u, z_u) and (x_v, y_v, z_v). The *normal* to S at (u,v) is the unit vector pointing at the direction of $(x_u, y_u, z_u) \times (x_v, y_v, z_v)$; it is therefore perpendicular to the tangent plane.

In the sequel, we shall use the fact that if C_1 and C_2 are curves which intersect on S, then the normal to S at their intersection point is a unit vector at the direction of the vector product of their tangent vectors. This holds unless these tangent vectors are parallel.

The intersection of S with any plane containing N is called a *normal section* of S. Note that the normal section is determined by a unit vector v in the tangent plane, which is the direction at which the plane containing N intersects the tangent plane. Thus, we may speak of a normal section at the direction v.

The curvature of a normal section is called the *normal curvature.* The maximal such curvature, k_1, and the minimal, k_2, are called the *principle curvatures* of S. Let us denote their directions by $\vec{k_1}$ and $\vec{k_2}$. It can be proved that they are orthogonal and that, if $v = \vec{k_1}\cos(\theta) + \vec{k_2}\sin(\theta)$, then the normal curvature at the direction v equals

$$k_1 \cos^2(\theta) + k_2 \sin^2(\theta) \tag{3}$$

The product $K = k_1 k_2$ is called the *Gaussian curvature*, and the mean $H = \frac{k_1+k_2}{2}$ is the *mean curvature*.

Suppose a curve C lies on the surface S. Then, if its curvature is κ_C, and the normal curvature of S at the direction of C's tangent vector is κ_S, then

$$\kappa_S = \kappa_C \cos(\theta) \tag{4}$$

where θ is the angle between N_S, the normal to S, and N_C, the normal to C.

3.3 Elimination Theory

Elimination theory is a branch of algebra which deals with eliminating variables from equations. It is especially useful for determining when a system of equations has a root. Let us start with the simplest case – two polynomials in one variable, $p = p_n x^n + p_{n-1}x^{n-1} + ... + p_0$, and $q = q_m x^m + q_{m-1}x^{m-1} + ... + q_0$.

To compute the *resultant* of p and q, one first constructs an $(n+m) \times (n+m)$ matrix as follows. Its first row consists of p's coefficients, followed by zeros. The second row is obtained by translating the first one to the right, etc. When this can be done no more, the same process is repeated for q's coefficients. The resultant is equal to the determinant of this matrix. For instance, the resultant of $ax^4 + bx^3 + cx^2 + dx + e$ and $Ax^3 + Bx^2 + Cx + D$ is the determinant of

$$\begin{pmatrix} a & b & c & d & e & 0 & 0 \\ 0 & a & b & c & d & e & 0 \\ 0 & 0 & a & b & c & d & e \\ A & B & C & D & 0 & 0 & 0 \\ 0 & A & B & C & D & 0 & 0 \\ 0 & 0 & A & B & C & D & 0 \\ 0 & 0 & 0 & A & B & C & D \end{pmatrix}$$

A basic result in elimination theory is that the resultant is equal to zero if p and q have a common root.

It is also possible to eliminate variables from systems of polynomials with more equations. For example, if we have three polynomial equations with two variables, there is an expression in the coefficients of these polynomials which is zero if the system has a solution. In general, elimination is a difficult problem, and it is not always possible to explicitly write down these expressions.

4 The General Method

In this section, a general overview of the method for deriving differential and semi-differential invariants for curves lying on surfaces is provided.

We wish to find conditions on the curvature and torsion of a curve C which will allow us to determine if it possibly lies on a certain geometric object $\mathcal{OBJ}$, which is described by a generic implicit equation, $P(x,y,z)=0$.

The method by which these conditions is derived proceeds as follows. First, we use the local canonical form to write down an expression for C in the vicinity of a point M we have measured on $\mathcal{OBJ}$; we also assume that we have measured κ, τ, and their derivatives, as well as the Frenet trihedron at M. These are all determined from the derivatives of C; so, if we have accurate measurements for C in the vicinity of M, we may directly calculate them. Since κ and τ do not depend on the pose of the C, we are allowed to translate and rotate $\mathcal{OBJ}$ – and the curve on it – thus obtaining a new curve $\mathcal{C}$. Denote the rotated and translated object by $\mathcal{OBJ}_{new}$.

Every condition on τ and κ we derive for $\mathcal{C}$ is, of course, also a condition for C. The reason we apply rigid transformations to $\mathcal{OBJ}$ is because these allow us to make assumptions on $\mathcal{C}$'s Frenet trihedron which result in simpler calculations; this will be explained in the sequel. Let $\mathcal{P}(x,y,z)$ be the implicit equation defining $\mathcal{OBJ}_{new}$.

Next, we substitute $\mathcal{C}$'s local canonical form into $\mathcal{P}(x,y,z)$; This results in a Taylor series in s. This series *has to be identically zero*, because $\mathcal{C}$ is contained in $\mathcal{OBJ}_{new}$, and, therefore, has to satisfy the equation which defines $\mathcal{OBJ}_{new}$. This gives us a set of equations – each for every coefficient in the Taylor series. Next, we eliminate from these equations everything *but* $\mathcal{C}$'s curvature and torsion. For one curve, we usually have to eliminate the Frenet trihedron. For two curves, we will show that the Frenet trihedrons are known and therefore need not be eliminated. In both cases, the elimination gives an expression that has to be zero; and

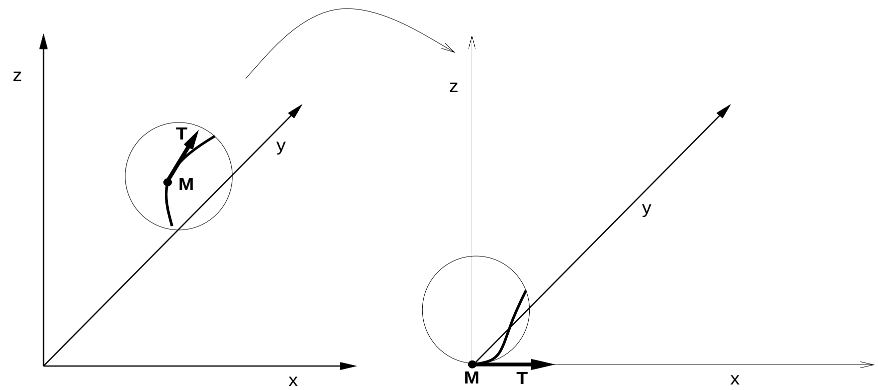


Figure 4: Rotating and translating the sphere.

this is the sought invariant.

We proceed to apply this paradigm to specific objects; first, the sphere is tackled.

5 The Case for a Sphere

In order to derive a differential invariant for a curve $c(s)$ to lie on a sphere, we need to use only the following part of c's local canonical form:

$$c(s) = c(0) + (s - \frac{\kappa^2 s^3}{6})T + (\frac{s^2\kappa}{2} + \frac{s^3\kappa'}{6})N - \frac{s^3}{6}\kappa\tau B + o(s^3) \tag{5}$$

Since translation and rotation do not change the curvature and torsion, we may assume, without loss of generality, that the point M, at which our measurements of κ and τ were taken, is at the origin, and that the sphere lies on the XY plane. Hence, the sphere's equation is

$$x^2 + y^2 + (z - R)^2 - R^2 = 0 \tag{6}$$

Let us also assume, without loss of generality, that the sphere had been rotated so that $T = (1, 0, 0)$ (see Figure 4).

Since N is a unit vector perpendicular to T, it has to be of the form $N = (0, \cos(\alpha), \sin(\alpha))$ for some α; also, $B = T \times N = (0, -\sin(\alpha), \cos(\alpha))$.

Note that the rigid transformation applied to the sphere has reduced the Frenet trihedron to a trihedron *depending only on the single parameter* α. This is important, because we have to eliminate the trihedron, in order to obtain a condition depending only on κ and τ; and, in general, the more variables we have to eliminate, the more equations are necessary, and there's a danger that the solution will be extremely complicated.

Substituting these T, N, B in Equation 5 gives the following expressions for the components of $c(s)$:

$$x(s) = s - \frac{s^3 k^2}{6}$$

$$y(s) = \frac{s^2 k \cos(\alpha)}{2} + \frac{s^3 (\kappa' \cos(\alpha) + k\tau \sin(\alpha))}{6}$$

$$z(s) = \frac{s^2 k \sin(\alpha)}{2} + \frac{s^3 (\kappa' \sin(\alpha) - k\tau \cos(\alpha))}{6}$$

Plugging these expressions into the sphere's equation (6) gives a Taylor series in s, which has to be identically zero, therefore all its coefficients are zero. The expression is rather complicated, so we don't write it down here; However, its constant and linear coefficients are identically zero, the coefficient of s^2 is

$$1 - k \sin(\alpha) R = 0 \tag{7}$$

The coefficient of s^3 is

$$\kappa' \sin(\alpha) R - k\tau \cos(\alpha) R = 0 \tag{8}$$

And, naturally, we have the equation

$$\sin^2(\alpha) + \cos^2(\alpha) - 1 = 0 \tag{9}$$

We may view these as algebraic equations, by treating $\sin(\alpha)$ and $\cos(\alpha)$ as algebraic variables. Then, from these three Equations (7,8,9), we may eliminate $\sin(\alpha)$ and $\cos(\alpha)$, to obtain the identity

$$R^2 = \frac{k^2 \tau^2 + (\kappa')^2}{k^4 \tau^2} \tag{10}$$

This gives us a differential invariant for a curve lying on a sphere; namely, the expression

$$\frac{k^2 \tau^2 + (\kappa')^2}{k^4 \tau^2}$$

has to be a constant. Note that we can immediately extract the sphere's radius.

It should be noted that this condition has been derived before, using other methods (see, for instance, [6], page 25). We have nonetheless decided that it's worthwhile to show how

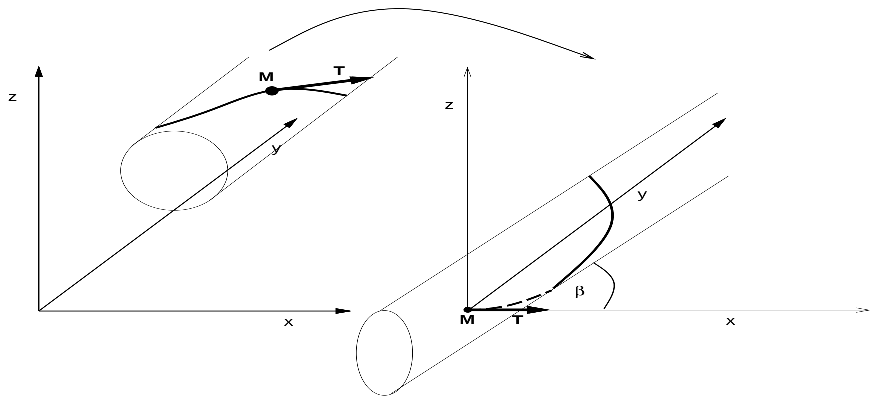


Figure 5: Rotating and translating the cylinder.

it is derived by using the local canonical form and elimination theory. This derivation will hopefully make it easier to follow the derivation of differential invariants for curves on the cylinder and cone, presented in the following sections.

6 The Case for a Cylinder

6.1 One Curve, Known Radius

We now proceed to derive differential invariants for a curve which lies on a cylinder. To the best of our knowledge, such invariants have not been derived before. The method is roughly the one used for the sphere, however, the mathematical details are considerably more complicated.

Given a point M on a curve which lies on a cylinder, we can assume without loss of generality that the cylinder had been translated and rotated so that M is at the origin, and the cylinder lies on the XY plane (recall that this does not alter the curvature and torsion). Let us further assume that it had been rotated at some angle β so that the tangent vector at M is aligned with the X-axis (see Figure 5).

Hence, $T = (1, 0, 0)$, and the cylinder's equation becomes

$$(x\cos(\beta) + y\sin(\beta))^2 + (z - R)^2 - R^2 = 0 \tag{11}$$

As for the sphere, it follows that $N = (0, \cos(\alpha), \sin(\alpha))$ for some α, and $B = (0, -\sin(\alpha), \cos(\alpha))$.

We now substitute these T, N, B in the local canonical form (2). This gives the following expressions for the components of $c(s)$:

$$x(s) = s - \frac{s^3\kappa^2}{6} - \frac{s^4\kappa\kappa'}{8} + \frac{s^5(-4\,\kappa\kappa'' - 3\,\kappa'^2 + \kappa^4 + \kappa^2\tau^2)}{120}$$

$$\begin{aligned} y(s) = {} & \frac{s^2\kappa\cos(\alpha)}{2} + \frac{s^3(\kappa'\ \cos(\alpha) + \kappa\tau\ \sin(\alpha))}{6} + \\ & \frac{s^4(\kappa''\ \cos(\alpha) + 2\,\kappa'\,\tau\ \sin(\alpha) - \kappa^3\cos(\alpha) + \kappa\tau'\ \sin(\alpha) - \kappa\tau^2\cos(\alpha))}{24} + \\ & \frac{s^5}{120}(\kappa'''\ \cos(\alpha) + 3\,\kappa''\,\tau\ \sin(\alpha) + 3\,\kappa'\,\tau'\ \sin(\alpha) - 3\,\kappa'\,\tau^2\cos(\alpha) - \\ & 6\,\kappa^2\kappa'\ \cos(\alpha) - \kappa^3\tau\ \sin(\alpha) + \kappa\tau''\ \sin(\alpha) - 3\,\kappa\tau\,\tau'\ \cos(\alpha) - \kappa\tau^3\sin(\alpha)) \end{aligned}$$

$$\begin{aligned} z(s) = {} & \frac{s^2\kappa\sin(\alpha)}{2} + \frac{s^3(\kappa'\ \sin(\alpha) - \kappa\tau\ \cos(\alpha))}{6} + \\ & \frac{s^4(\kappa''\ \sin(\alpha) - 2\,\kappa'\,\tau\ \cos(\alpha) - \kappa^3\sin(\alpha) - \kappa\tau'\ \cos(\alpha) - \kappa\tau^2\sin(\alpha))}{24} + \\ & \frac{s^5}{120}(\kappa'''\ \sin(\alpha) - 3\,\kappa''\,\tau\ \cos(\alpha) - 3\,\kappa'\,\tau'\ \cos(\alpha) - 3\,\kappa'\,\tau^2\sin(\alpha) - \\ & 6\,\kappa^2\kappa'\ \sin(\alpha) + \kappa^3\tau\ \cos(\alpha) - \kappa\tau''\ \cos(\alpha) - 3\,\kappa\tau\,\tau'\ \sin(\alpha) + \kappa\tau^3\cos(\alpha)) \end{aligned}$$

Plugging these into the cylinder's equation (11) gives, as before, a Taylor series in s which has to be identically zero. This expression is huge and we do not write it down here; we need only the coefficients of the powers of s between 0 and 5.

The coefficients of the constant and linear terms are identically zero.

For the other terms, we obtain the following expressions, after substituting $\cos(\alpha) = x, \sin(\alpha) = y, \cos(\beta) = z, \sin(\beta) = w$:

For the coefficient of s^2

$$-2\,\kappa yR + 2\,z^2 \ = \ 0 \tag{12}$$

For the coefficient of s^3

$$6\,zw\kappa x - 2\,\kappa'\,yR + 2\,\kappa\tau\,xR \ = \ 0 \tag{13}$$

For the coefficient of s^4

$$8\,zw\kappa\tau\,y - 2\,\kappa''\,yR + 4\,\kappa'\,\tau\,xR + 2\,\kappa^3 yR + 2\,\kappa\tau'\,xR + 2\,\kappa\tau^2 yR - \\ 6\,\kappa^2 x^2 z^2 + 8\,zw\kappa'\,x - 8\,z^2\kappa^2 + 6\,\kappa^2 \;=\; 0 \tag{14}$$

In the sequel, it will be beneficial to use a simplified version of (14). Note that we can subtract from (14) the product of (12) by an appropriate constant, and eliminate the coefficient of z^2 in (14) (it already has an yR term, so we are not adding anything). Similarly, we can subtract from the new equation an appropriate multiple of (13), to remove from it the term with the monomial xzw – also, without adding anything new, as the set of monomials of (14) contains that of (13). After grouping, we can write the simplified (14) as

$$A_0 + A_1 x^2 z^2 + A_2 xR + A_3 yR + A_4 yzw \;=\; 0 \tag{15}$$

Note that we can easily compute the A_i's as functions of κ and τ. Hence, (15) is equivalent to (14), but much simpler. This will turn out to be useful.

For the coefficient of s^5, we obtain the equation

$$-2\,\kappa'''\,yR - 20\,\kappa^2 y\tau\,xz^2 - 20\,\kappa\kappa'\,x^2 z^2 - 30\,zw\kappa^3 x - 10\,zw\kappa\tau^2 x + 10\,zw\kappa\tau'\,y + \\ 20\,zw\kappa'\,\tau\,y + 10\,zw\kappa''\,x - 2\,\kappa^3\tau\,xR + 6\,\kappa''\,\tau\,xR - 30\,z^2\kappa\kappa' + 6\,\kappa'\,\tau^2 yR + \\ 12\,\kappa^2\kappa'\,yR + 6\,\kappa'\,\tau'\,xR + 2\,\kappa\tau''\,xR - 2\,\kappa\tau^3 xR + 20\,\kappa\kappa' + 6\,\kappa\tau\,\tau'\,yR \;=\; 0 \tag{16}$$

in addition we have

$$x^2 + y^2 - 1 \;=\; 0 \tag{17}$$

$$z^2 + w^2 - 1 \;=\; 0 \tag{18}$$

Assume now that the radius R is known. In that case, we have to eliminate x, y, z, w from Equations (12,13,14,17,18) (note that we need at least five equations in order to eliminate four unknowns). All our attempts to directly do this, using various packages for symbolic computations, have failed; however, it is possible to proceed as follows. First, solve the system consisting of the four simplest equations (12,13,17,18). Then, substitute the solution into (14).

Using the Maple symbolic computation program, it was possible to find a solution for Equations (12,13,17,18). This solution uses an auxiliary polynomial we denote by $p_1(\delta)$:

$$p_1(\delta) = 81\,\kappa^8\delta^8R^2 + (18\,R^2{\kappa'}^2\kappa^4 - 18\,\kappa^6\tau^2R^2 + 81\,\kappa^6 - 162\,\kappa^8R^2)\delta^6 + 36\,\kappa^4R\delta^5\kappa'\,\tau + (-81\,\kappa^6 + 81\,\kappa^8R^2 - 36\,R^2{\kappa'}^2\kappa^4 + \kappa^4\tau^4R^2 + 2\,\kappa^2\tau^2R^2{\kappa'}^2 + R^2{\kappa'}^4 + 18\,\kappa^6\tau^2R^2)\delta^4 - 36\,\kappa^4R\delta^3\kappa'\,\tau + (18\,R^2{\kappa'}^2\kappa^4 - 2\,\kappa^2\tau^2R^2{\kappa'}^2 - 2\,R^2{\kappa'}^4)\delta^2 + R^2{\kappa'}^4$$

Denote by q a root of $p_1(\delta)$. Then, the solution of (12,13,17,18) equals

$$z = R\sqrt{-\frac{9\,\kappa^4q^4 - q^2\kappa^2\tau^2 + q^2{\kappa'}^2 - 9\,\kappa^4q^2 - {\kappa'}^2}{q(9\,q\kappa^2 + 2\,\kappa'\,\tau\,R)}}$$

$$w = \frac{-9\,\kappa'\,R\kappa^4q^4 + \kappa'\,Rq^2\kappa^2\tau^2 + {\kappa'}^3Rq^2 - 9\,\kappa'\,R\kappa^4q^2 - {\kappa'}^3R + 9\,\kappa^4\tau\,q^3}{3q^2(9\,q\kappa^2 + 2\,\kappa'\,\tau\,R)\kappa^2\sqrt{-\frac{9\,\kappa^4q^4 - q^2\kappa^2\tau^2 + q^2{\kappa'}^2 - 9\,\kappa^4q^2 - {\kappa'}^2}{q(9\,q\kappa^2 + 2\,\kappa'\,\tau\,R)}}}$$

$$y = -\frac{R(9\,\kappa^4q^4 - q^2\kappa^2\tau^2 + q^2{\kappa'}^2 - 9\,\kappa^4q^2 - {\kappa'}^2)}{q(9\,q\kappa^2 + 2\,\kappa'\,\tau\,R)\kappa}$$

$$x = q$$

Substituting these expressions into (14) and simplifying, we obtain the following identity

$$(18\,k^6\tau^2R^2 - 45\,R^2{k'}^2k^4 + 27\,R^2\kappa''\,k^5 + 162\,k^8R^2 - 81\,k^6)q^6 + (27\,k^5\tau'\,R - 54\,\tau\,\kappa'\,Rk^4)q^5 + (-36\,k^6\tau^2R^2 - k^2\tau^2R^2{\kappa'}^2 + 6\,\tau\,\kappa'\,k^3\tau'\,R^2 + 3\,{\kappa'}^2R^2\kappa''\,k - 2\,k^4\tau^4R^2 - 5\,R^2{\kappa'}^4 + 72\,R^2{\kappa'}^2k^4 + 162\,k^6 - 3\,\tau^2R^2\kappa''\,k^3 - 162\,k^8R^2 - 27\,R^2\kappa''\,k^5)q^4 + 90\,\tau\,\kappa'\,Rk^4q^3 + (-3\,{\kappa'}^2R^2\kappa''\,k + 6\,R^2{\kappa'}^4 + 9\,k^2\tau^2R^2{\kappa'}^2 - 27\,R^2{\kappa'}^2k^4)q^2 - R^2{\kappa'}^4 = 0$$

Let us denote this polynomial by $p_2(q)$.

Now, we know that $p_1()$ and $p_2()$ *must have a common root*; therefore, their resultant must be zero. This resultant is, therefore, an invariant for a curve lying on a cylinder.

Recalling the definition of the resultant of two polynomials (Section 3.3), we can write down the resultant of $p_1()$ and $p_2()$. It is a determinant whose elements depend on the curvature and torsion; if the curve lies on a cylinder, this determinant has to be zero, and this is an invariant for a curve lying on a cylinder.

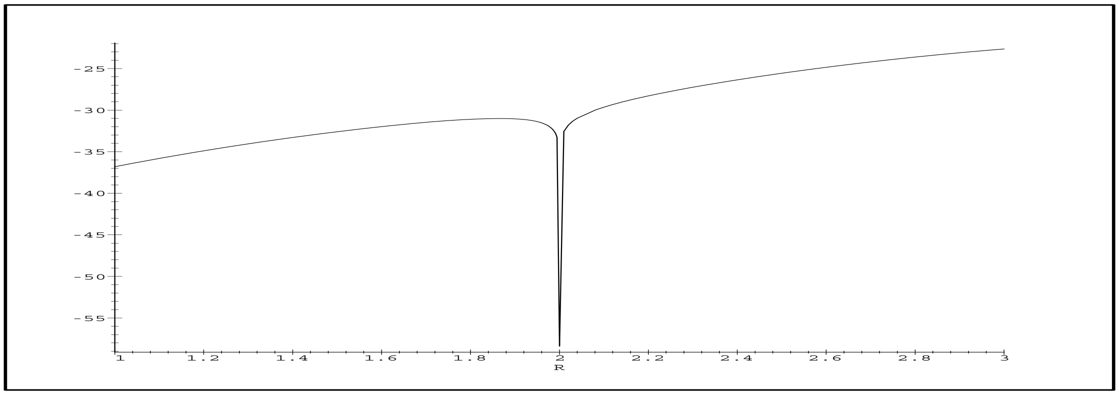


Figure 6: Logarithm of the resultant of $p_1()$ and $p_2()$, as a function of R, for a point sampled from a curve on a cylinder of radius 2.

6.2 One Curve, Unknown Radius

6.2.1 Numerical Search for the Correct Radius

Suppose we do not know the radii of the cylinders in the data-base.

There are two ways to proceed. We can simply follow the trivial observation that, if we substitute the correct R into $p_1()$ and $p_2()$, we will get two polynomials whose resultant is zero. We can therefore conduct a simple, one-dimensional search for R which minimizes this resultant.

Experience has shown us that this simple numerical algorithm works quite well. For example, in Figure 6, a plot for the logarithm of the resultant, for values of κ and τ measured on a curve on a cylinder with a radius of 2, is displayed. We can clearly see a strong minimum at the correct radius.

6.2.2 Solve for the Correct Radius

The second method for the case in which the radius is unknown is to eliminate R, x, y, z, w from Equations 12,13,15,16,17,18. This can be done by solving Equations 12,13,15,17,18, and substituting the solution in Equation 16; if this gives zero, it means that these six equations have a common solution, which is a necessary condition for the curve to lie on a cylinder. This is why it was important to define Equation 15, the simplified version of (14); we could not find a reasonable solution with (14). However, it turns out that Equations 12,13,15,17,18 do have a relatively simple solution, expressed as follows. x is the root of the following equation:

$$
\begin{aligned}
&81\,\kappa^8 A_1{}^2\delta^{12} + (-162\,A_1{}^2\kappa^8 - 54\,\tau\,A_4\,A_1\,\kappa^7 + 162\,\kappa^8 A_1\,A_0 + 162\,A_3\,A_1\,\kappa^7)\delta^{10} + \\
&(108\,\tau\,A_4\,A_1\,\kappa^7 - 54\,\kappa^7 A_0\,A_4\,\tau + 81\,A_2{}^2\kappa^6 + 162\,\kappa^7 A_0\,A_3 - 18\,A_0\,\tau^2\kappa^6 A_1 + 81\,\kappa^8 A_0{}^2 - \\
&54\,\kappa'\,A_4\,A_2\,\kappa^5 - 54\,\tau\,A_4\,A_3\,\kappa^6 - 324\,A_3\,A_1\,\kappa^7 + 9\,\tau^2 A_4{}^2\kappa^6 + 9\,\kappa'^2 A_4{}^2\kappa^4 - 324\,\kappa^8 A_1\,A_0 + \\
&81\,A_1{}^2\kappa^8 + 18\,\kappa^4 A_1\,\kappa'^2 A_0 + 81\,A_3{}^2\kappa^6)\delta^8 + (162\,\kappa^8 A_1\,A_0 - 18\,A_0\,\tau^2\kappa^5 A_3 + \\
&18\,\kappa^4 A_0{}^2\kappa'^2 + 108\,\tau\,A_4\,A_3\,\kappa^6 - 162\,A_3{}^2\kappa^6 - 18\,\tau^2 A_4{}^2\kappa^6 + 18\,A_0\,\tau^2\kappa^6 A_1 + 6\,\kappa^3 A_4\,\tau\,\kappa'^2 A_0 - \\
&36\,\kappa^4 A_1\,\kappa'^2 A_0 - 18\,A_0{}^2\tau^2\kappa^6 + 108\,\kappa^7 A_0\,A_4\,\tau - 36\,\kappa^4\kappa'\,\tau\,A_0\,A_2 + 18\,\kappa^3 A_3\,\kappa'^2 A_0 + \\
&108\,\kappa'\,A_4\,A_2\,\kappa^5 - 324\,\kappa^7 A_0\,A_3 + 6\,A_0\,\tau^3\kappa^5 A_4 + 162\,A_3\,A_1\,\kappa^7 - 27\,\kappa'^2 A_4{}^2\kappa^4 - 54\,\tau\,A_4\,A_1\,\kappa^7 - \\
&81\,A_2{}^2\kappa^6 - 162\,\kappa^8 A_0{}^2)\delta^6 + (36\,\kappa^4\kappa'\,\tau\,A_0\,A_2 - 54\,\kappa'\,A_4\,A_2\,\kappa^5 - 12\,\kappa^3 A_4\,\tau\,\kappa'^2 A_0 + \\
&27\,\kappa'^2 A_4{}^2\kappa^4 + 18\,A_0{}^2\tau^2\kappa^6 + A_0{}^2\tau^4\kappa^4 + 9\,\tau^2 A_4{}^2\kappa^6 - 54\,\kappa^7 A_0\,A_4\,\tau + \kappa'^4 A_0{}^2 + 81\,\kappa^8 A_0{}^2 + \\
&162\,\kappa^7 A_0\,A_3 + 81\,A_3{}^2\kappa^6 + 2\,A_0{}^2\tau^2\kappa^2\kappa'^2 - 54\,\tau\,A_4\,A_3\,\kappa^6 - 6\,A_0\,\tau^3\kappa^5 A_4 - \\
&36\,\kappa^3 A_3\,\kappa'^2 A_0 + 18\,A_0\,\tau^2\kappa^5 A_3 - 36\,\kappa^4 A_0{}^2\kappa'^2 + 18\,\kappa^4 A_1\,\kappa'^2 A_0)\delta^4 + (-2\,A_0{}^2\tau^2 k^2\kappa'^2 + \\
&6\,\kappa^3 A_4\,\tau\,\kappa'^2 A_0 + 18\,\kappa^4 A_0{}^2\kappa'^2 + 18\,\kappa^3 A_3\,\kappa'^2 A_0 - 9\,\kappa'^2 A_4{}^2\kappa^4 - \\
&2\,\kappa'^4 A_0{}^2)\delta^2 + \kappa'^4 A_0{}^2
\end{aligned}
$$

Note that this is really a sixth-degree equation, as only even powers of δ appear. After x is solved for, we can easily extract y from Equation 17. Then, after substituting the known values of x and y in Equations 12,13,18 we can solve for the remaining unknowns – w, z and R:

Define ϵ to be

$$\kappa^2\tau^2x^2 - 2\,k\tau\,xy\kappa' + y^2\kappa'^2 + 9\,\kappa^4x^2y^2$$

and then

$$\{w = -\frac{Root(\epsilon\,\delta^2 - 9\,\kappa^4x^2y^2)(k\tau\,x - y\kappa')}{3\,\kappa^2xy}, z = Root(\epsilon\,\delta^2 - 9\,\kappa^4x^2y^2), R = \frac{9\,\kappa^3yx^2}{\epsilon}\} \quad (19)$$

(by *Root* of an equation, we mean the root of the equation when viewed as an equation in δ). The equations in (19) are trivial to solve and involve only taking square roots.

The reader may ask why we did not apply this trick to simplify the solution of the equations for the case in which the radius is known. If we had done that, it would not have been possible to obtain a function of κ and τ alone; the x would still have been there! And, as long as it is there, we cannot find a condition on κ and τ, as desired, but only a condition on x, κ and τ.

There is also a direct solution to the system of equations 12,13,15,17,18, in which all the unknowns – R, x, y, z, w – are written in terms of κ, τ, and their derivatives; however, that expression is truly horrendous, covering three entire pages when written in small format! For all practical purposes, it is better to use the solution above, which first extracts x and y and then solves for the other unknowns.

6.2.3 Comparison of Methods

While the second method is straightforward and does not require any search (as opposed to the first method), it has the drawback of requiring the fifth derivative of the curve, which appears in Equation 16 (note that calculating the third derivative of the curvature and the second derivative of the torsion requires the fifth derivative of the curve). The first method requires a numerical search for the correct radius, but uses only the fourth derivative of the curve. Depending on how accurate the measurements are, one may opt for using the first or the second method.

6.3 The case for a Cylinder with Two Intersecting Curves and Unknown Radius

Suppose we have two curves on the cylinder, intersecting at a point M. For instance, one can design a sensory robot to traverse a point twice, in different directions. Another possible source is an intersecting pattern of structured light rays. It turns out that a particularly simple invariant can be written in this case.

We refer to the curves as "first" and "second" (it makes no difference which is which, of course). As noted in Section 3.2, two intersecting curves on the surface allow us to compute its normal $\mathcal{N}$ (denoted this way to prevent confusion with N, the normal to a curve). We may, as before, translate and rotate the cylinder so that the intersection point M is in the origin, the cylinder lies on the XY plane, and the tangent vector of the first curve equals (1,0,0). The difference is that now, as opposed to when we only had a single curve, we know the normal N and the binormal B of the new curves; this is because now we know that the rotation and translation not only move M to the origin and align the tangent to the first curve with $(1, 0, 0)$, they also align $\mathcal{N}$ with $(0, 0, 1)$. Let us look at the triplet (T, N, B) for the first curve (before the rotation). We can calculate the inner products (N,T) and $(N,\mathcal{N})$. These inner products do not change after the rotation of the curve; if N is rotated into N_{new}, then, since T is rotated into $(1,0,0)$, we have the equality $(N_{new},(1,0,0))=(N,T)$, and, since $\mathcal{N}$ is rotated into $(0,0,1)$, we have the equality $(N_{new},(0,0,1))=(N,\mathcal{N})$. Since N_{new} is a unit vector, we can recover it; and, since we know the tangent and normal of the new curve, we know its binormal, which is equal to their vector product. Following a similar argument, we also know the Frenet trihedron of the (new) second curve .

As before, let β denote the angle in which the cylinder is aligned relative to the XY plane. Let us denote the tangent, normal and binormal of the first curve by $(1,0,0)$, $(0,\cos(\alpha),\sin(\alpha))$ and $(0,-\sin(\alpha),\cos(\alpha))$, and those of the second curve by $(T_1,T_2,0)$, (N_1,N_2,N_3), (B_1,B_2,B_3) (remember that all these coordinates are now known). Note that the z-coordinate of both tangents has to be zero, as they are both in the tangent plane which, after the rigid transformation, is the XY plane.

Substituting these expressions into the local canonical form, then into the cylinder's equation, and equating coefficients to zero, results in the following equations (K is the curvature of the second curve):

For the coefficient of s^2, first curve, we have

$$2\,z^2-2\,k\sin(\alpha)R \;=\; 0 \tag{20}$$

For the coefficient of s^3, first curve, we have

$$6\,zwk\cos(\alpha)-2\,\kappa^{'}\,\sin(\alpha)R+2\,k\tau\,\cos(\alpha)R \;=\; 0 \tag{21}$$

For the coefficient of s^2, second curve, we have

$$2\,{T_2}^2-2\,KN_3\,R+2\,z^2{T_1}^2-2\,{T_2}^2z^2+4\,zT_1\,wT_2 \;=\; 0 \tag{22}$$

Also

$$z^2+w^2-1 \;=\; 0 \tag{23}$$

(where, as before, $\cos(\beta)=z,\sin(\beta)=w$).

Eliminating w,z and R from (20,21,22,23) results in the identity

$$9\,{T_1}^2\sin(\alpha)^2\cos(\alpha)^2\kappa^4-6\,\kappa^3\cos(\alpha)^2\sin(\alpha)\tau\,T_2\,T_1-9\,\kappa^3\cos(\alpha)^2\sin(\alpha)KN_3+\kappa^2\tau^2{T_2}^2\cos(\alpha)^2+$$

$$6\,\kappa^2 T_1\,T_2\,\kappa^{'}\sin(\alpha)^2\cos(\alpha) - 2\,\kappa^{'}\,\tau\,{T_2}^2\sin(\alpha)\cos(\alpha)\kappa + (\kappa^{'})^2{T_2}^2\sin(\alpha)^2 = 0$$

(remember that α is known, and does not have to be eliminated).

And this is an invariant for two intersecting curves, which can be used to test whether they lie on a cylinder. The invariant depends on the curvature and torsion of one curve, and the curvature of the other; therefore, it does not require any derivatives of order higher than three.

7 The Case For a Cone with Two Intersecting Curves

We have not addressed the problem of finding invariants for a cone using a single curve; because a cone has more degrees of freedom than a sphere or a cylinder, this would necessitate using the sixth derivative of a curve to express such an invariant.

We proceed to show how two intersecting curves yield an invariant for the cone. We will not go into all the details, as the method resembles the one used for a cylinder with two intersecting curves.

First, the cone is rotated and translated so that its apex is at the origin, and the point of intersection of the two curves, M, lies on the XY plane, which is also the tangent plane at M. Then, it is rotated in the XY plane so that the tangent vector of the first curve is $(1,0,0)$. As for the cylinder, we can extract the tangent, normal, and binormal vectors to the two curves at their new location; denote the normal to the first curve at M by $(0,\cos(\beta),\sin(\beta))$. Note that now M does not lie at the origin, but at an (unknown) distance of y_0 from it. The (unknown – as for the cylinder) rotation angle of the cone in the XY plane is denoted by α, and the (unknown) rotation angle around the Y axis (Figure 7) is denoted θ; this is just half of the cone's opening angle.

It is then a trivial matter to write down the equation of the rotated and translated cone, and to substitute into it the local canonical forms of the two curves. As before, the coefficients of the two resulting Taylor series have to be zero, resulting in the following equations. Let K and $\mathcal{T}$ denote the curvature and torsion of the second curve, and T_1 etc. the components of its Frenet trihedron vectors. S stands for $\sin(\beta)$, C for $\cos(\beta)$, x for $\cos(\theta)$, y for $\sin(\theta)$, z for $\cos(\alpha)$, w for $\sin(\alpha)$ (remember that S and C are known, and do not have to be solved for):

For the coefficient of s^2, first curve, we have

$$2\,x^2z^2 + 2\,xkSy_0\,y \;=\; 0 \tag{24}$$

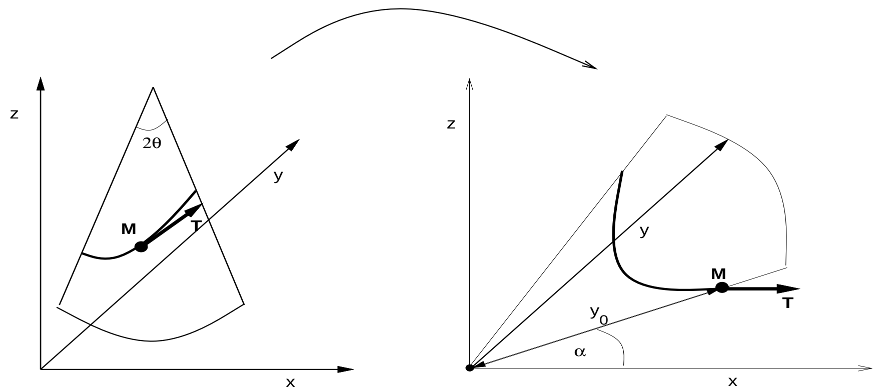


Figure 7: Rotating and translating the cone.

For the coefficient of s^3, first curve, we have

$$-2\,xk\tau\,Cy_0\,y + 6\,yxwkS + 2\,x\kappa^{'}\,Sy_0\,y - 6\,x^2zwkC \;=\; 0 \tag{25}$$

Which can be written more compactly as

$$A_1\,x^2zw + A_2\,xyy_0 + A_3\,xyw \;=\; 0 \tag{26}$$

For the coefficient of s^2, second curve, we have

$$2\,x^2z^2{T_1}^2 - 4\,x^2zwT_1\,T_2 + 2\,x^2{T_2}^2 - 2\,x^2{T_2}^2z^2 + 2\,xKN_3\,y_0\,y \;=\; 0 \tag{27}$$

Just as for the cylinder, we can subtract from (27) appropriate multiples of (24) and (25), and obtain the simpler form

$$A_4\,x^2 + A_5\,xyy_0 + A_6\,x^2zw \;=\; 0 \tag{28}$$

For the coefficient of s^3, second curve, we have

$$\begin{aligned} &6\,yxzKN_3\,T_2 + 6\,yxwKN_3\,T_1 + 6\,x^2T_2\,KN_2 - 6\,x^2T_2\,KN_2\,z^2 - 2\,xK\mathcal{T}\,B_3\,y_0\,y + \\ &2\,xK1\,N_3\,y_0\,y - 6\,x^2zwT_1\,KN_2 - 6\,x^2zwKN_1\,T_2 + 6\,x^2z^2T_1\,KN_1 \;=\; 0 \end{aligned} \tag{29}$$

Which, as before, can be reduced to

$$A_7\,xyw + A_8\,xyz + A_9\,x^2 + A_{10}\,xyy_0 \;=\; 0 \tag{30}$$

Note that the A_i's can be readily computed from the known quantities – the curvature, torsion, and Frenet trihedron of the two curves.

We also have the equations

$$x^2 + y^2 - 1 \;=\; 0 \tag{31}$$

and

$$w^2 + z^2 - 1 \;=\; 0 \tag{32}$$

It is possible to eliminate $\{x, y, z, w, y_0\}$ from these six equations (24,26,28,30,31,32), and obtain

$$\begin{aligned}
&-2\, SkA_4{}^3 A_3\, A_{10}\, A_2\, A_7 - A_4{}^2 A_2{}^2 A_7{}^2 A_5 + 2\, S\kappa A_4{}^2 A_2\, A_7\, A_5\, A_3\, A_9 - S\kappa A_6\, A_4{}^2 A_2{}^2 A_7\, A_8 - \\
&S^2\kappa^2 A_6{}^2 A_3{}^2 A_9{}^2 A_5 - A_3{}^2 A_9{}^2 A_5{}^3 + S\kappa A_4\, A_3{}^2 A_9{}^2 A_5{}^2 - S^2\kappa^2 A_6{}^2 A_4\, A_3{}^2 A_9\, A_{10} - \\
&S^2\kappa^2 A_6{}^3 A_3\, A_9\, A_2\, A_8 + 2\, S^2\kappa^2 A_6\, A_4{}^2 A_2\, A_8\, A_3\, A_9 - S^2\kappa^2 A_4{}^2 A_1{}^2 A_8{}^2 A_5 - \\
&S^3\kappa^3 A_6\, A_4{}^2 A_1{}^2 A_7\, A_8 - 2\, S^2\kappa^2 A_4{}^2 A_1\, A_5\, A_8\, A_3\, A_9 + 2\, S^3\kappa^3 A_6\, A_4{}^2 A_1\, A_7\, A_3\, A_9 + \\
&S\kappa A_6\, A_4{}^2 A_3\, A_{10}\, A_2\, A_8 - 2\, S\kappa A_4{}^2 A_3{}^2 A_{10}\, A_5\, A_9 + S^3\kappa^3 A_4{}^3 A_1{}^2 A_8{}^2 + S^2\kappa^2 A_6\, A_4{}^2 A_1\, A_8{}^2 A_2 + \\
&S^3\kappa^3 A_4{}^3 A_1{}^2 A_7{}^2 + S\kappa A_4{}^3 A_2{}^2 A_8{}^2 + S^3\kappa^3 A_6{}^2 A_4\, A_3{}^2 A_9{}^2 + 2\, A_4\, A_3{}^2 A_9\, A_5{}^2 A_{10} - \\
&2\, A_4\, A_3\, A_9\, A_5{}^2 A_2\, A_7 - 3\, S\kappa A_6\, A_4\, A_3\, A_9\, A_5\, A_2\, A_8 + S\kappa A_4{}^3 A_3{}^2 A_{10}{}^2 + 2\, S\kappa A_4\, A_3\, A_9\, A_5{}^2 A_1\, A_8 + \\
&2\, S^2\kappa^2 A_4{}^3 A_1\, A_8\, A_3\, A_{10} - 2\, S\kappa A_4{}^2 A_1\, A_8\, A_3\, A_{10}\, A_5 + S^2\kappa^2 A_6{}^2 A_4\, A_3\, A_9\, A_2\, A_7 + S\kappa A_4{}^3 A_2{}^2 A_7{}^2 - \\
&S^3\kappa^3 A_6{}^2 A_4\, A_3\, A_9\, A_1\, A_8 - S\kappa A_6{}^2 A_3\, A_9\, A_2\, A_7\, A_5 + S\kappa A_6\, A_3\, A_9\, A_1\, A_5{}^2 A_7 + \\
&S\kappa A_6{}^2 A_3{}^2 A_9\, A_{10}\, A_5 + S^2\kappa^2 A_6{}^2 A_3\, A_9\, A_1\, A_5\, A_8 - 2\, S^2\kappa^2 A_4{}^2 A_1{}^2 A_7{}^2 A_5 + \\
&S^2\kappa^2 A_6\, A_4{}^2 A_1\, A_7{}^2 A_2 - S^2\kappa^2 A_6\, A_4{}^2 A_1\, A_7\, A_3\, A_{10} - A_4{}^2 A_3{}^2 A_{10}{}^2 A_5 - 3\, S^2\kappa^2 A_6\, A_4\, A_1\, A_7\, A_3\, A_9\, A_5 + \\
&2\, A_4{}^2 A_3\, A_{10}\, A_2\, A_7\, A_5 - S\kappa A_6\, A_4\, A_1\, A_7{}^2 A_2\, A_5 + S\kappa A_6\, A_4\, A_1\, A_7\, A_3\, A_{10}\, A_5 - \\
&S^2\kappa^2 A_6{}^2 A_4\, A_1\, A_7\, A_2\, A_8 + S^2\kappa^2 A_6\, A_4\, A_1{}^2 A_7\, A_5\, A_8 + S\kappa A_4\, A_1{}^2 A_7{}^2 A_5{}^2 = 0
\end{aligned}$$

And this is an invariant for two curves on a cone. It depends on the curvature and torsion of the two curves; therefore, it does not require any derivatives of order higher than three. This is an invariant for two curves on a cone.

8 Numerical Computation of Derivatives

The algorithms suggested here require computing the derivatives of a curve in 3D space. The problem of computing high-order derivatives from discrete data was addressed in [25]. The derivatives at each point are calculated by convolving appropriate differentiation filters with the given curve. One way of deriving such filters is based on fitting high-order polynomials to the data curve and differentiating the polynomial. We do not need to do the fitting for each actual curve; it is only done in deriving the filters.

In deriving the filters, the data curve f is approximated by a linear combination of orthogonal polynomials of orders $0, \ldots, l$:

$$F_l(-x) = w(x) \sum_{i=0}^{l} a_i P_i(x)$$

where $P_i(x)$ are polynomials which are orthonormal with respect to a weight function $w(x)$. The coefficients a_i are determined by the condition that the polynomial fits the curve in the sense of (weighted) least squares. It can be proved that if the curve f is a polynomial of order up to l, than the above filter yields an exact k-th derivatives when the coefficients a_i are:

$$a_i = P_i^{(k)}(0)$$

In practice, good results are obtained for any reasonably smooth f (not only polynomials), as long as the the order l of the filter is larger than the desired order k of the derivative. However, a high l requires a filter with a wide support.

Discrete versions of this method on a finite interval are described in detail in [15]. In particular, the Krawtchouk and the discrete Chebyshev polynomials were studied and closed form formulas for them were given up to fifth order. However, it was shown in [25] that continuous polynomials, defined on a finite interval, are just as effective but much simpler to calculate. Good results were obtained using the Legendre and continuous Chebyshev polynomials.

For example, see Figure 8 for the derivative of the spherical invariant (Equation 10), for the curve plotted in Figure 3, when noise of variance equal to 5 percent of the distance between the points was added to it. Derivatives were computed using the method described in [25]. The derivative is relatively small for the part of the curve that lies on the sphere ($0 \leq t \leq 1$), and significantly changes when the curve crosses over to the cylinder (at $t = 1$). Note that computing the invariant's derivatives requires the first derivative of the torsion and the second derivative of the curvature, that is, the fourth derivative of the curve.

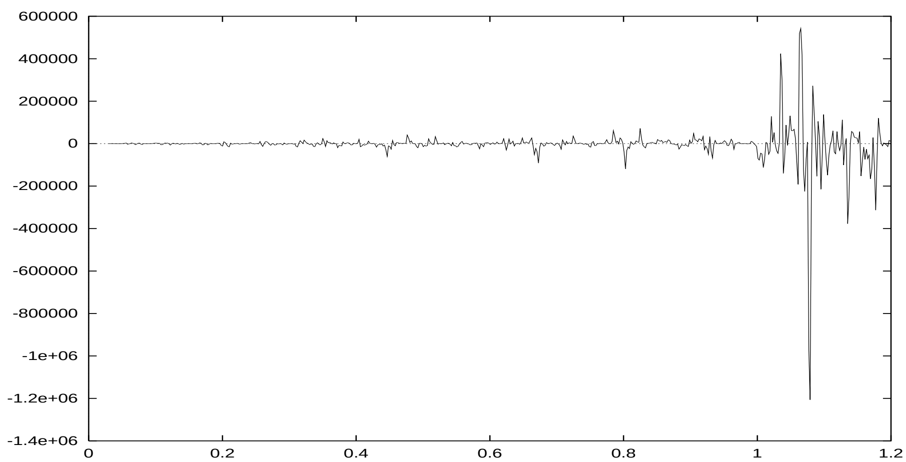


Figure 8: Derivative of spherical invariant (Equation 10) for the curve in Figure 3, after noise had been added to it. Note change in derivative when the curve crosses over from the sphere to the cylinder (at $t = 1$).

9 Semi-Differential Invariants

In this section we study curve invariants which use only curvature (this requires computing only the first and second derivatives of the curve). We also assume that the only primitives the recognition system may encounter are spheres, cylinders, cones, and tori. When the information from one point is not enough to uniquely determine the object, we will use an additional point or two on the curve to help disambiguate the object.

9.1 Object Recognition from Two Intersecting Curves

Given two intersecting curves C_1 and C_2, we extract $T_1, N_1, B_1, \kappa_{c1}, T_2, N_2, B_2, \kappa_{c2}$ at the intersection point M. These are the Frenet trihedrons and the curvature for both curves respectively. Recall that N_S, the normal to the surface at M, equals $T_1 \times T_2$.

For each curve we compute θ, the angle between N_S and the curve's normal. The surface normal curvature equals $\kappa_{N_S} = \kappa_C \cos(\theta)$, and $\kappa_{N_S}(\beta) = \kappa_1 \sin^2(\beta) + \kappa_2 \cos^2(\beta)$, where κ_1, κ_2 are the principal curvatures for the surface at M, and β is the angle between the tangent to the curve and $\vec{\kappa_2}$, the second principal direction.

Given two curves we have two equations for the surface normal curvature, with three unknowns – κ_1, k_2, and β:

$$\begin{aligned} \kappa_{N_S 1} &= \kappa_1 \sin^2(\beta) + \kappa_2 \cos^2(\beta) \\ \kappa_{N_S 2} &= \kappa_1 \sin^2(\beta + \phi) + \kappa_2 \cos^2(\beta + \phi), \end{aligned} \tag{33}$$

where ϕ is the angle between T_1 and T_2. Usually, it is impossible to solve such a system; however, if we know in advance that the geometric primitives can only be spheres, cylinders, cones, and tori, it is possible to identify them and extract their parameters.

Sphere

In this case $\kappa_1 = \kappa_2$ and consequently $\kappa_{N_S 1} = \kappa_{N_S 2}$. For all other objects (cylinders, cones, and tori) the two principle curvatures are not equal; therefore, two distinct normal curvatures are identical only in the degenerate case in which the angles between the curves' tangents and $\vec{\kappa_1}$ are equal. Therefore, if the surface normal curvatures corresponding to the two curves are equal, we can assume with high probability that we are dealing with a sphere.

The sphere's radius is then $R = 1/\kappa_{N_S}$, and its center is at $M + RN_S$. Using an additional point (Section 8.2), we can determine if the object is indeed a sphere.

Cylinder

If the given object is a cylinder, its parameters can be recovered as follows. As $\kappa_1 = 0$, the surface normal equations are reduced to two equations with two unknowns. Solving them, we can recover κ_2 and the principal directions $\vec{\kappa_1}, \vec{\kappa_2}$. The cylinder's radius is $R = \frac{1}{\kappa_2}$, and the orientation of its axis is $\vec{\kappa_1}$. A point on the axis is:

$$C = M + RN_S.$$

It is important to note that this does not prove that the object is a cylinder. That has to be verified using an additional point on the curve (see Section 8.2).

Cone

Assume the object is a cone. As for the cylinder, $\kappa_1 = 0$, and we can recover κ_2 and the principal directions $\vec{\kappa}_1, \vec{\kappa}_2$. The radius of the cone at M is $R = \frac{1}{\kappa_2}\cos(\alpha)$, where α is the cone's opening angle. The apex is located at $M + \vec{\kappa_1}\frac{\cot(\alpha)}{\kappa_2}$, and the axis orientation is $\vec{\kappa_1}\cos(\alpha) + \vec{\kappa_2}\sin(\alpha)$. α can be determined from an additional point on the curve (see Section 8.2).

Torus

From (33) we cannot recover the torus, because the number of unknowns is three. We will parameterized our solution as a parameter of β. For a given β, we can recover κ_1, κ_2. The values of κ_1, κ_2 change on the torus as a function of γ, the angle between the major radius of the torus, R, and the vector to the current point on the torus. κ_1, κ_2 are given as a function of γ:

$$\kappa_1 = \frac{-\cos(\gamma)}{R + r\cos(\gamma)} \qquad \kappa_2 = \frac{1}{r}$$

where R and r are the major and minor radii of the torus respectively (see Figure 9).

Given $\kappa_1, \kappa_2, \gamma$ we can recover R, r as follows:

$$R = -(\frac{1}{\kappa_1} - \frac{1}{\kappa_2})\cos(\gamma) \qquad r = \frac{1}{\kappa_2}$$

The orientation of the torus, N_t, can be recovered by : $N_t = N_S\sin(\gamma) + \vec{\kappa}_2\cos(\gamma)$. The center of the torus is then at:

$$C = M + N_t r + (N_S\cos(\gamma) - \vec{\kappa}_2\sin(\gamma))R$$

β and γ can be determined by an additional point on the curve (see Section 8.2).

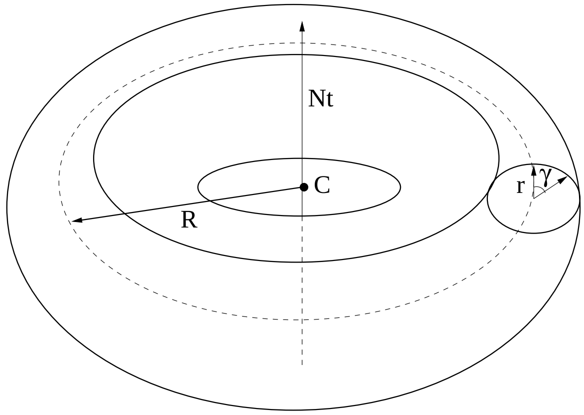


Figure 9: Torus.

9.2 Verification Using an Additional Point

The hypothesis about an object and its parameters can be verified by an additional point on one of the curves. For the hypothesis to be correct, several constraints must be satisfied.

- The point M must lie on the surface. This means that if P is the object's implicit equation, $P(M) = 0$.
- T_C, the curve's tangent, must be orthogonal to N_S at the point. Thus
$$N_S \cdot T_C = 0.$$
- If θ is the angle between N_S and N_C, then $\kappa_S = \kappa_C \cos(\theta)$ (see Equation 4), where the value of κ_S is determined by the principal curvatures κ_1 and κ_2 and the angle between them and T_C.

Therefore, each additional point yields three additional equations which have to be satisfied. These equations can be used to verify hypotheses or to determine the value of unknown parameters.

If the additional points are not on a curve, and we don't have any differential properties associated with them, we still have the first condition (they have to satisfy the surface equation). In that case, we will need more points; this is a typical tradeoff for semi-differential invariants.

9.3 Object Recognition from One Curve

When two intersecting curves are given, we are able to recover N_S and thus we know the angle θ between N_S and N_C. When we are given only one curve, θ is an unknown parameter which has to be recovered.

Sphere

In this case $\kappa_1 = \kappa_2$, and consequently $\kappa_{N_S} = 1/R$. For every value of θ, the surface normal and the sphere's radius are determined as follows, where (T_C, N_C, B_C) are the Frenet trihedron of the curve:

$$N_S = \cos(\theta)N_C + \sin(\theta)B_C \quad R = \frac{1}{\kappa_C \cos(\theta)}$$

From that we recover the center of the sphere,

$$M + RN_S = M + \frac{N_S}{\kappa_C} + \tan(\theta)B_C \tag{34}$$

Thus we have a family of possible spheres, parameterized by θ.

Given additional points, we can proceed as follows: either substitute them at the (hypothesized) sphere's equation, or (if they are on a curve) use the verification method described in Section 8.2. Alternatively, given two points on a curve, applying Equation 34 to both of them results in four linear equations in $\cos(\theta_1), \tan(\theta_1), \cos(\theta_2)$, and $\tan(\theta_2)$. The solution is verified by checking if the two angles satisfy

$$\tan(\theta_i) = \frac{\sqrt{1 - \cos^2(\theta_i)}}{\cos(\theta_i)}.$$

Cylinder

In the case of the cylinder we know that $\kappa_1 = 0$ and $\kappa_2 = \frac{1}{R}$. Given a point M_1 on the curve, the two unknowns are θ_1 and β_1. When they are given, the cylinder is uniquely defined. Note that $\vec{\kappa_1}$ is the axis of the cylinder, so it has to be the same for every point on the cylinder. We will now use these facts to define R and $\vec{\kappa_1}$ the axis of the cylinder as functions of θ_1 and β_1 (see Equations 33, 4):

$$R = \frac{\cos^2(\beta_1)}{\kappa_{C1} \cos(\theta_1)}.$$

$$N_S = \cos(\theta_1)N_C + \sin(\theta_1)B_C$$

$$\vec{\kappa_1} = T_C \sin(\beta_1) + (T_C \times N_S)\cos(\beta_1) \tag{35}$$

And a point on the axis is:

$$C_1 = M_1 + RN_{S1}$$

Given an additional point, its β_2 and θ_2 can be recovered as follows:

$$\beta_2 = \arcsin(T_{C2} \cdot \vec{\kappa_1}), \quad \theta_2 = \arccos(\frac{\cos^2(\beta_2)}{\kappa_{C2} R}).$$

From them we can recover the point on the axis C_2 closest to the second point, and both points must lie on the cylinder's axis, which is parallel to $\vec{\kappa_1}$; therefore,

$$(C_1 - C_2) \times \vec{\kappa_1} = 0,$$

which gives us two equations in two unknowns, which can be solved for the values of θ_1 and β_1.

These two points give the equation of the cylinder that passes through them and satisfies the given constraints. In addition, from (35) $\beta_2 = \arccos((T_{C2} \times N_{S2}) \cdot \vec{\kappa_1})$, which gives an additional constraint to verify that this is indeed a cylinder with the computed parameters.

Cone

The case of the cone is similar to the cylinder but slightly more complicated. Given two points on a curve we would like to find the angles $\theta_1, \beta_1, \theta_2$, and β_2. These angles parameterize the local surface structure of the two points. At first we will exploit the fact that the line from the point on the surface in the direction of $\vec{\kappa_1}$ must pass through the tip of the cone. Thus we have a constraint that the two such lines of the two points must intersect. The point $C = P + \kappa_2 N_S$ lies on the central axis of the cone. Therefore we two additional constraints which are due to the fact that C_1, C_2, and the tip of the cone lie on the same line. Finally, the angle of the cone α must be the same for both surface points. As α is the angle between $\vec{\kappa_1}$ and the axis of the cone, we can write an additional constraint enforcing the uniqueness of α. With the four above mentioned constraints we can recover the values of the unknown angles and recover the shape of the object.

As in the cylinder, these two points give the equation of the cone that passes through them and satisfies the given constraints. However, an additional point is needed to verify that this is indeed the real object.

Torus

In order to be able to recover the seven parameters of the torus, we parameterize them by four local parameters of one point. The parameters are θ, β, κ_1, and γ. As described above these four parameters are enough to describe the torus. In order to recover those parameters we need two additional points because each point yields three constraints. Thus using three points we can recover the shape of the torus and verify that the object is indeed a torus.

9.3.1 Experimental Results

The algorithm for a single curve has been tested on real data received from the Brown & Sharpe Company using their coordinate measuring machines (Figure 1). The data is a curve measured on a cylinder. For each point on the curve T_C, N_C, B_C, and κ_C are estimated. Using the algorithm described above, the problem is reduced to solving for $\cos(\theta_1)$ and $\cos(\beta_1)$, where all other parameters are expressed as functions of these unknown values. The correct values must satisfy four equations and have to satisfy the constraints that the absolute values of the cosine and sine of the various angles must be less than 1. The values of the unknowns are found using non-linear least squares optimization techniques. In this case we use the Levenberg-Marquardt procedure of the MINPACK library [16].

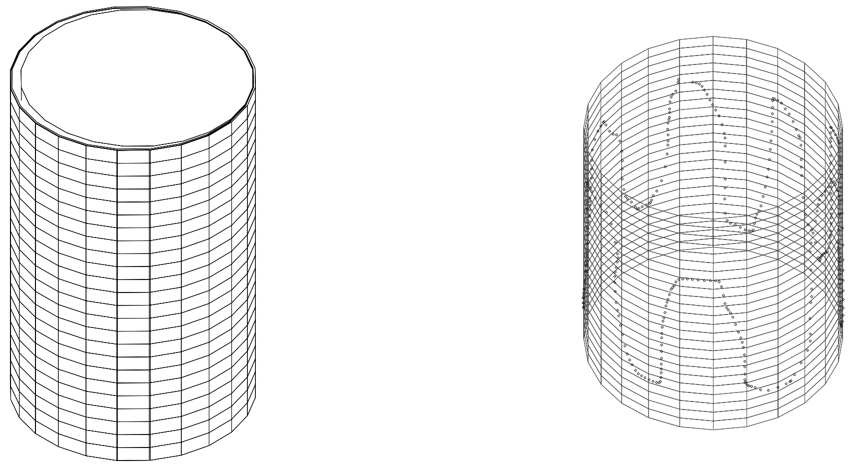
We chose at random 200 pairs of points and ran the minimization procedure on them using several initial conditions for the minimization. Even though the data is noisy, most pairs of points yielded results close to the correct shape. The results were sorted according to the least-squares error (LSE) of the four equations. We trace the five cylinders with the smallest LSE in Figure 10(a). One of these results and the original data are shown in 10(b). It is important to note that only the data on the two points and their derivatives mentioned above was used to recover the shape of the cylinder. Additional points can then be used, if desired, to get a better estimate for the shape.

10 Conclusions

A novel method to recognize some surfaces, given curve(s) on them, was presented. It proceeds by using invariants which are computed on curves, but which supply information on the type of surfaces the curve can possibly lie on.

The method can use 3D curves derived from stereo and structured light; it is particularly useful when given the output of measuring devices which produce such curves (for instance, sensory robots and coordinate measuring machines).

The main advantage of the proposed method compared to algebraic methods is in its local nature, which enables it to segment and recognize curves (and the surfaces they lie on),



(a) (b)

Figure 10: (a) The five recovered cylinders with the lowest LSE. (b) The recovered shape of the cylinder and the data points.

even if the curves lie on more than one geometric primitive. Also, it necessitates a far smaller number of curve points than the algebraic method, for recognizing a single primitive.

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