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On Mobile Robot Localization From Landmark Bearings

Ilan Shimshoni

***Abstract*—This paper deals with the problem of robot localization from noisy landmark bearings measured by the robot. We present a new localization method which is based on linear constraints, one due to each bearing measurement. This linear system can be solved at low computational cost but yields not very accurate results. Therefore, we transform the system to an equivalent linear system which yields virtually optimal results at a small fraction of the cost of a nonlinear optimization method, which usually achieves the optimal result. Experimental results showing the quality of the results and the low computational cost are presented.**

***Index Terms*—Angle measurements, landmarks, robot localization.**

I. Introduction

When mobile robots or automated guided vehicles (AGVs) are moving in their workspaces, one of the basic problems which needs to be solved is for the robot to know its position and orientation in the plane as accurately as possible. One of the standard methods for performing this task is to put landmarks in known locations in the workspace. In any place in the workspace, the robot is able to measure the bearings to a sufficient number of these landmarks. Using three or more such measurements, the robot is able to estimate its position and orientation in the plane [1]–[4].

The most widely used method for computing this estimate is a geometric method based on the idea that the angle between two such bearing measurements yields the constraint that the robot's position is limited to lie on a circle. Adding an additional bearing measurement yields two more circles whose intersection is the desired location. The problem begins when there are errors in the measurements. To overcome this problem, more than three landmarks are placed in the workspace, and all the measured bearings have to be used to get the optimal estimate for the robot's position. The geometric method does not lend itself naturally to more than three bearing measurements. Thus, costly nonlinear minimization techniques with their known problems have to be employed.

We would like, therefore, to take a different approach, which has been presented in [5], but without dealing with the accuracy problems.

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We will use each bearing measured to define a constraint on the position and orientation of the robot. These will be linear constraints. When given three or more bearing measurements, we have that number of constraints, and using numerical linear algebra techniques [singular value decomposition (SVD)] we are able to solve this linear system and find the best solution for this system very efficiently.

The problem is, however, that when there is uncertainty in the bearing measurements, the solution to this linear system is not the optimal solution to the original problem. We, therefore, present several transformations of the linear system yielding more accurate results. All these transformations have negligible computational cost, but improve the accuracy of the results considerably. In the final algorithm, we present a simple optimization procedure whose computational cost is twice or three times more than the previous algorithm, which further improves the accuracy. The quality of the results are essentially equal to the optimal results which can be achieved by nonlinear optimization but at a much higher cost.

For many applications, it is very important not only to estimate the position and orientation of the robot, but also to estimate the accuracy of the estimate by estimating its covariance matrix. Without these estimates, it is impossible to guarantee robustness of motion planning algorithms which use the localization algorithm. The covariance matrix is estimated by simply conducting several dozen experiments on input with synthetically added noise. This method is different than the method presented in [6], which computes a worst-case estimate of the uncertainty region when bounded error is assumed. We compute the estimated accuracy over all locations in several examples of workspaces.

This can then be used to plan robust safe and fast motion plans for the robot as shown in [7]–[13], and to plan optimally the placement of the landmarks in the environment when given the accuracy requirements on the robot's localization at certain positions in the workspace [14]. Another extension of this work could be to deal with the case in which the locations of the landmarks are unknown. This problem has been addressed in [5], and more generally in works dealing with simultaneous localization and map-building problems (SLAM) [15], [16].

The paper continues as follows. In Section II, we review the geometric method for estimating the robot's position and orientation and also show the algebraic method. In Section III, we describe the requirements needed to produce more accurate results and present transformations to the derived constraints which improve the accuracy of the results. These algorithms are then run and experimental results are presented. In Section IV, we conclude the paper by discussing applications for this algorithm and further research directions.

II. Basic Methods

A. Problem Statement

We are given a set of n landmarks whose positions $\mathbf{P}_i \in \mathbf{R}^2$ are known. The robot is at an unknown position $\mathbf{P}$ and orientation θ. The bearing measurements by the robot to these landmarks are β_i.

The problem is illustrated in Fig. 1. Here, three two-dimensional (2-D) points $\mathbf{P}_1$, $\mathbf{P}_2$, and $\mathbf{P}_3$ are viewed by the robot in directions β_1, β_2, and β_3, respectively. The goal is to find the position $\mathbf{P}$ and orientation θ of the robot.

B. Geometric Method

The geometric method is based on the following constraint. When given two landmarks in positions $\mathbf{P}_i$ and $\mathbf{P}_j$ and whose measured bearings are β_i and β_j, there exists a circle going through $\mathbf{P}_i$ and $\mathbf{P}_j$ such that the angle difference between the directions of the two bearings is $\beta_i - \beta_j$ (see Fig. 2). When given three such landmarks, the intersection of the three circles, which are due to the three pairs of measurements,

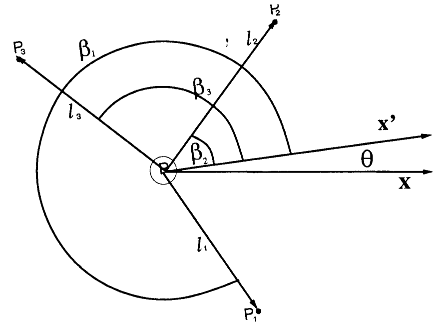

Fig. 1. Illustration of a planar mobile robot which detects three known landmarks in directions $\beta_1, \beta_2, \beta_3$. The goal is to find the robot's position and orientation in the plane.

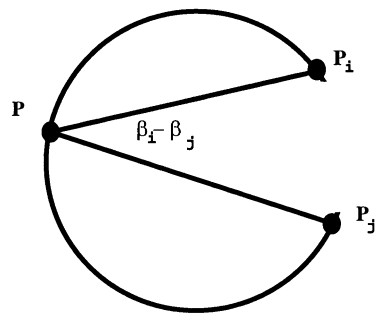

Fig. 2. Given two landmarks $\mathbf{P}_i$ and $\mathbf{P}_j$ whose bearing measurements are β_i and β_j, respectively. The possible robot positions lie on a circle illustrated in the figure.

yield the unique solution to the problem. There exists an inherent ambiguous situation when the robot's position is cocircular with the positions of the three landmarks. In this case, the robot's position can only be estimated to lie anywhere on the circle on which the landmarks lie, and additional landmark bearing measurements are needed to disambiguate the robot's position.

C. Algebraic Method

In this method, we will look at the problem as a coordinate transformation problem from the global coordinates $\mathbf{P}_i$ to the robot coordinates $\mathbf{M}_i$. The change is due to the robot's position $\mathbf{P}$ and orientation θ. Applying the transformation to a single bearing measurement yields the following:

$$\mathbf{M}_i = l_i(\cos\beta_i, \sin\beta_i) = R(\mathbf{P}_i - \mathbf{P}) = R\mathbf{P}_i + \mathbf{T} \tag{1}$$

where R is a planar rotation matrix by θ, $\mathbf{T} = -R\mathbf{P}$, and l_i is the unknown distance from $\mathbf{P}$ to $\mathbf{P}_i$. This can be written as the following two equations:

$$\begin{aligned} l_i \cos\beta_i &= (\cos\theta, \sin\theta) \cdot \mathbf{P}_i + \mathbf{T}_x \\ l_i \sin\beta_i &= (-\sin\theta, \cos\theta) \cdot \mathbf{P}_i + \mathbf{T}_y. \end{aligned} \tag{2}$$

l_i can be eliminated, yielding

$$(\cos\theta, \sin\theta) \cdot \mathbf{P}_i + \mathbf{T}_x = ((-\sin\theta, \cos\theta) \cdot \mathbf{P}_i + \mathbf{T}_y) \cdot \cot\beta_i. \tag{3}$$

This is a homogeneous linear equation in four unknowns

$$(\mathbf{P}_{ix} - \mathbf{P}_{iy}\cot\beta_i, \mathbf{P}_{iy} + \mathbf{P}_{ix}\cot\beta_i, 1, -\cot\beta_i) \cdot \begin{pmatrix} \cos\theta \\ \sin\theta \\ \mathbf{T}_x \\ \mathbf{T}_y \end{pmatrix} = 0.$$

We will denote by A_i the coefficient vectors which are computed from $\mathbf{P}_i$ and $\cot\beta_i$, and by W the vector of the unknown parameters. Taking equations for three such points yields the following:

$$AW = \begin{pmatrix} A_1^T \\ A_2^T \\ A_3^T \end{pmatrix} W = 0.$$

In order to find W, we apply SVD to A [17] (Ch. 2.6). This decomposes A into

$$A = USV^T$$

where U and V are orthogonal matrices, and S is a diagonal matrix whose diagonal elements are nonnegative numbers sorted by size. The last (fourth) vector of matrix V which corresponds to the smallest element in the diagonal of S is the solution to the problem.

As this is an homogeneous set of equations, the solution for W is determined up to a scale factor. This scale factor can be determined by exploiting the fact that W_1 and W_2 are $\cos\theta$ and $\sin\theta$, respectively, and should, therefore, satisfy $W_1^2 + W_2^2 = 1$. Thus, W is recovered. However, also $-W$ could be the solution. To verify which of them is the correct solution, we substitute the solution into the initial set of (2), and check if the right-hand sides of the equations have the same sign as the $\cos\beta_i$ and $\sin\beta_i$, whose sign is known but was lost in the derivation step from (2) to (3). Only one of the W's will satisfy these constraints.

If there are additional bearing measurements, each of them will add an additional row to A. Then the solution mentioned above will yield the optimal least-squares solution to this linear system, i.e.,

$$\arg\min_W \|AW\|^2 \text{ subject to } \|W\|^2 = 1.$$

III. Improved Linear Systems

Using SVD to solve our problem is very appealing because it is very efficient and has a linear computational complexity in the number of landmarks. The problem is that the solution obtained by SVD is not the optimal solution for the original problem we were trying to solve. The optimal solution we are seeking is the position $\mathbf{P}$ and rotation angle θ such that the bearings measured from that position and orientation $\hat{\beta}_{i(\mathbf{P},\theta)}$ are closest to the β_i's we measured in the experiment, and accounting for the fact that the variance of each measurement is σ_i^2. These requirements yield

$$\arg\min_{\mathbf{P},\theta} \sum_i \frac{\left(\beta_i - \hat{\beta}_{i(\mathbf{P},\theta)}\right)^2}{\sigma_i^2}. \tag{4}$$

For our linear system to yield close to optimal results, we should try to make it fulfill the following three requirements. The first requirement is that in a total least-squares problem (like the one we are solving), each entry in the matrix A should have the same variance [18] (Ch. 12.3). This variance is due to measurement errors. The second requirement is that

$$|A_i^T W|^2 = \frac{|\beta_i - \hat{\beta}_{i(\mathbf{P},\theta)}|^2}{\sigma_i^2}. \tag{5}$$

Thus, the solution will satisfy (4). That is, the error (as measured in standard deviations σ_i) in each measurement β_i will have the same effect on the solution. The last requirement is that each of the entries of the matrix A be independent of the others. This requirement can not be fulfilled for entries in the same row of the matrix, as they are all functions of a certain β_i, but it is fulfilled for entries from different rows as we assume that the measurements are independent.

We will first deal with the first requirement. Looking at the definition of A_i, it is obvious that this requirement is not fulfilled. The variance of each element $A_{i,j}$ is approximately $((\partial A_{i,j}/\partial\beta_i)\sigma_i)^2$. Thus, looking at

$$\frac{\partial A_i}{\partial \beta_i} = \left(\mathbf{P}_{iy}(\cot\beta_i^2 + 1), -\mathbf{P}_{ix}(\cot\beta_i^2+1), 0, (\cot\beta_i^2+1)\right).$$

We can see that the variance values are very nonuniform, both within a row of A and between the rows. The third element has variance 0 and the others are multiples of $-(\cot\beta_i^2 + 1)$, which can be unlimited in magnitude. Thus, a small measurement error can cause errors in different magnitudes for β_i's of different values.

The first step is to make the variance bounded. We will multiply each row by $\sin\beta_i$, yielding

$$A_i = (\mathbf{P}_{ix}\sin\beta_i - \mathbf{P}_{iy}\cos\beta_i, \mathbf{P}_{iy}\sin\beta_i + \mathbf{P}_{ix}\cos\beta_i, \sin\beta_i, -\cos\beta_i). \tag{6}$$

This makes the variance bounded

$$\frac{\partial A_i}{\partial \beta_i} = (\mathbf{P}_{iy}\sin\beta_i + \mathbf{P}_{ix}\cos\beta_i, \mathbf{P}_{iy}\cos\beta_i - \mathbf{P}_{ix}\sin\beta_i, \cos\beta_i, \sin\beta_i).$$

The variance is bounded by $|\mathbf{P}_i|^2\sigma_i^2$ for the first two entries, but by σ_i^2 for the second two entries. To make the first two entries have similar values to the other two, we apply a transformation to the world coordinates of the landmark positions $\mathbf{P}_i$. Once the solution is found, the inverse transformation will be applied to it, yielding the position in the original world-coordinate system. This transformation is very similar to the one used in [19] and for the same reason, which is to improve the quality of the result obtained from solving a linear system. The problem addressed in [19] was, of course, a different one.

Thus, we compute the centroid $\mathbf{C}$ of the $\mathbf{P}_i$'s, and $SZ = \max_i |\mathbf{P}_i - \mathbf{C}|$ and apply the following transformation to the $\mathbf{P}_i$'s:

$$\mathbf{P}'_i = \frac{(\mathbf{P}_i - \mathbf{C})}{SZ}.$$

As a result of this transformation, $|\mathbf{P}'_i| \le 1$. Now the variances of all the entries have similar magnitudes. Applying these transformations to the original set of equations has a negligible computational cost, but improves the quality of the estimates considerably.

The last step in our algorithm is to ensure that the (5) is enforced. This constraint is equivalent to

$$\frac{\partial A_i W}{\partial \beta_i} = \frac{1}{\sigma_i}.$$

However, W is unknown. Therefore, we run SVD on the matrix A we just obtained and use the W obtained to compute

$$d_i = \frac{\partial A_i W}{\partial \beta_i}\sigma_i.$$

We then multiply each row A_i by $1/d_i$ and reapply the SVD on the new weighted version of A. This operation is repeated two or three times until the process converges. Thus, at a cost two or three times as much as the original computation, another improvement is achieved.

The algorithm can also be run incrementally. Given an estimate for W due to a previous running of the algorithm and dead-reckoning, and a single bearing measurement, a new estimate of W can be obtained by projecting W to the closest point which satisfies the new constraint of type (6). If variance estimates of the previous estimate of W and the current measurement are available, a W on the line between the previous estimate and closest point mentioned above can be chosen, yielding the most probable estimate for W. This algorithm is not as accurate as the algorithm which deals with all the constraints simultaneously, but can be used when not all bearing measurements are available at the same time.

A. Experimental Results

To test our algorithms we have implemented them using Matlab. To demonstrate the improvement achieved by the transformation steps, we

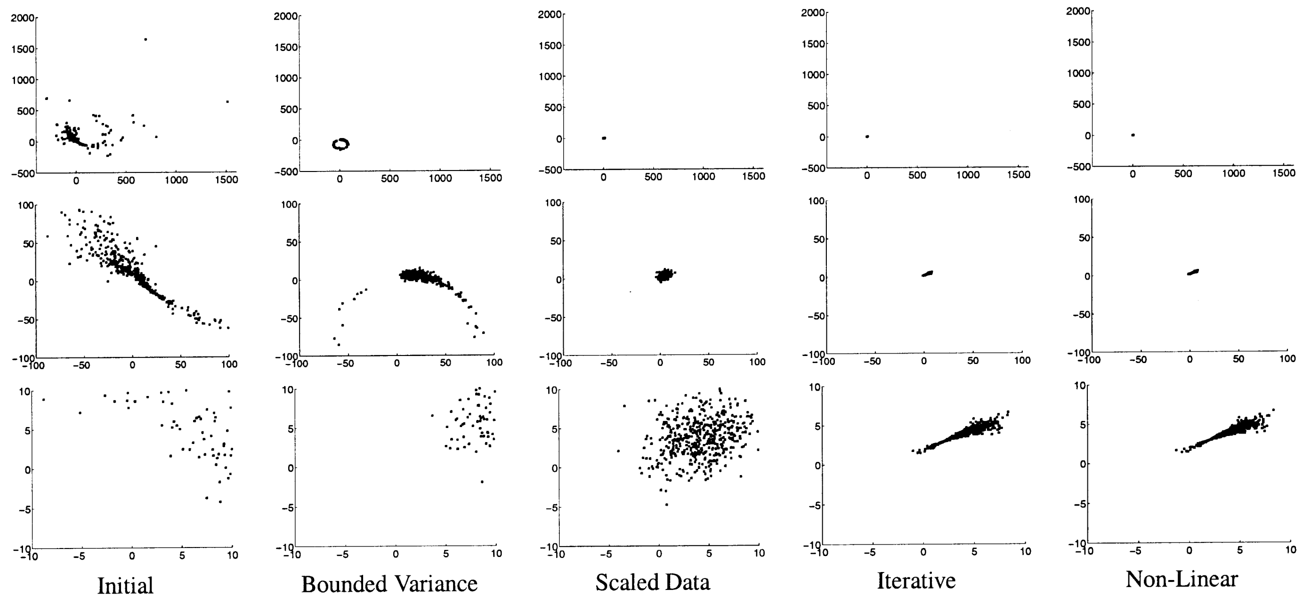

Fig. 3. For each of the five algorithms, we present the distribution of the results obtained from running it on 500 experiments. The correct answer is (4, 4) the landmarks lie within a circle of radius 100, and the measurement error is normally distributed with standard deviation of 5°. The results are plotted three times in different scales.

TABLE I
FOR EACH ALGORITHM THE TABLE SHOWS THE MEAN POSITION OBTAINED BY THE ALGORITHM [SHOULD BE (4,4)], THE VARIANCE IN THE x AND y COORDINATES, AND THE AVERAGE RUNNING TIME IN SECONDS FOR RUNNING EACH EXPERIMENT. THE QUALITY OF THE RESULTS IMPROVE FROM ONE ALGORITHM TO THE NEXT. THE FOURTH ALGORITHM GETS APPROXIMATELY THE SAME RESULTS AS THE NONLINEAR OPTIMIZATION ALGORITHM BUT AT 1% OF THE COMPUTATIONAL COST!!!. THUS IT SHOULD BE USED INSTEAD OF THE NONLINEAR OPTIMIZATION

Method	Mean Position	Variance	Running Time
Initial Algorithm	(18.04,35.108)	(14733,12844)	0.0084476
Bounded Variance	(20.594,-1.7789)	(355,639.2)	0.0065014
Scaled data	(4.3223,3.8965)	(7.6949,6.6746)	0.0070153
Iterative weighted rows	(3.9996,3.9624)	(2.7276,0.70346)	0.011813
Non-Linear Optimization	(3.9307,3.9347)	(2.7204,0.7094)	**1.4277**

ran an experiment in which several hundred sets of measurements with random noise were given to the different variants of the algorithm and a nonlinear optimization algorithm. The difference in the measurement sets were only due to the random noise. The obtained robot positions computed by the different algorithms are presented in Fig. 3, drawn to the same scale. As the difference in the quality of the different algorithms is so significant, we had to plot each of the results at three different scales so the results of the different algorithms can be appreciated. The reduction in the scattering of the results demonstrates the quality of the last two algorithms. The results are also presented in Table I, showing the mean and variance of the results of the different algorithms and their running times. The difference in variance between the first and last algorithm is a factor of 5400 (factor of the standard deviations is 73.6)!!! Our final algorithm achieves results which are nearly identical to those of the nonlinear optimization technique but at 1% of the running time!

We also tested the convergence rate of the iterative algorithm. Out of 10 000 random experiments performed with this algorithm, it converged after two applications of SVD in 92% of the time, three applications of SVD in 8% of the time, and only once with four applications of SVD. The reason for this fast convergence rate is that the initial result achieved after the first application of SVD is so close to the final solution (as can be seen in the results shown above) that usually no more iterations are needed, and that in all cases, the algorithm converges due to the accuracy of this initial result.

The accuracy of the estimates (the covariance matrix) can be also estimated directly by estimating the derivatives of the estimating function with respect to the input and output parameters. The details of this method are presented in [20] and [21].

Now that we have chosen an algorithm, we would like to show some results of running it in several environments. In each environment, the landmarks have been placed in certain positions, and we measure the accuracy of the estimate in each position in the workspace by repeating the experiment several dozen times and assessing the accuracy as the square root of the determinant of the covariance matrix or its trace. The values are displayed on a log scale. In the first example, shown in Fig. 4, the landmarks are arranged in two circles of the same radius, one with ten landmarks, and the other with five. The most accurate results are achieved near the landmarks. The results become less accurate as we approach the centers of the circles. When we move away from the circle, the accuracy again is reduced. Obviously, the circle with more landmarks yields better results. Had there been only one circle in the workspace, very inaccurate results would have been obtained close to the circle due to the inherent ambiguity mentioned above.

In the second example shown in Fig. 5, we positioned the landmarks randomly. The best results are obtained close to two landmarks which

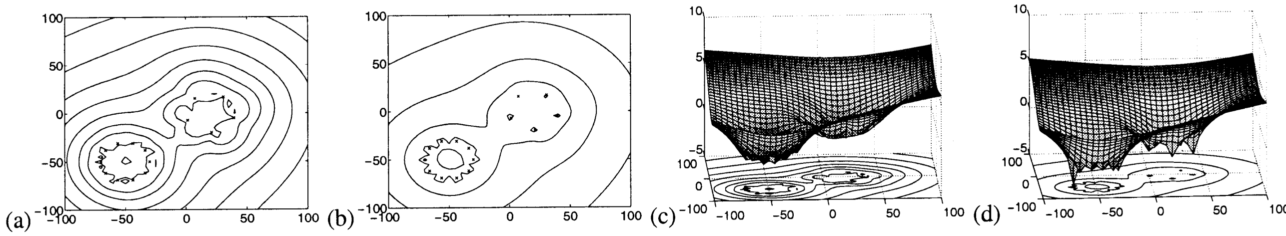

Fig. 4. Scene with two circles of landmarks with ten and five landmarks each, landmarks are marked with x's. (a) Contour map of the log of the trace of the covariance map. (b) Contour map of the log of the determinant of the covariance map. (c) and (d) Surfaces of these two uncertainty functions. Best results obtained near the landmarks of the circle with more landmarks.

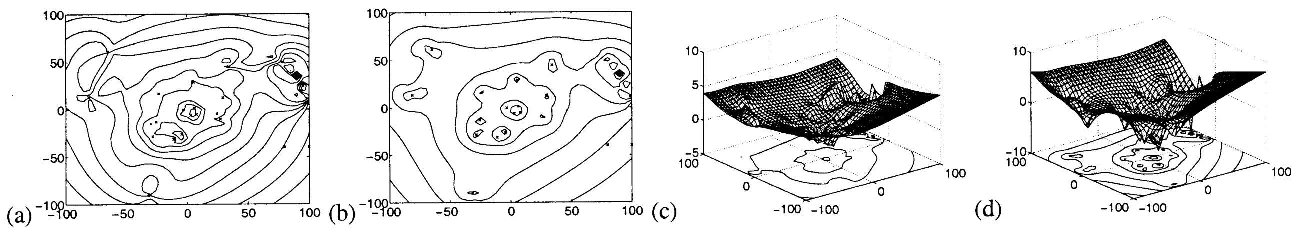

Fig. 5. Scene with a set of randomly placed landmarks. (a) Contour map of the log of the trace of the covariance map. (b) Contour map of the log of the determinant of the covariance map. (c) and (d) Surfaces of these two uncertainty functions. The localization algorithm does best close to two very close landmarks and does also well next to four landmarks a bit further apart.

were positioned very close to each other, and another accurate position was determined near a cluster of four landmarks.

IV. Applications and Future Directions

In conclusion, we have presented a new efficient localization algorithm for a mobile robot from noisy bearing measurements. We presented our algebraic solution and explained why several transformations should be applied to the linear system of equations in order to improve the accuracy of the solution, and showed experimentally that these transformations indeed improve the accuracy, yielding results which are very close to the optimal results, which usually require nonlinear optimization techniques. Our final algorithm is also very efficient, running more than a hundred times faster than the nonlinear optimization technique. We therefore recommend that our algorithm be used for robot localization from landmark bearing measurements.

The analysis presented here can be used to develop more accurate algorithms and study others. For example, in studying the algorithm to solve the problem in computer vision of estimating the fundamental matrix [19]. We can see that the scaling of the input that was performed there causes the variances between the different entries in the coefficient matrix to be more similar, improving the quality of the results. However, as one of the columns in the matrix is a constant 1 (variance 0), not all entries in the matrix have similar variances, and thus, the solution that this algorithm achieves is not as close to the optimal solution as we achieved. This is obviously due to the difference between the problems solved.

The main use of this algorithm is to accurately recover the position and orientation of the robot at very low computational cost. As shown above, we can also estimate the accuracy of our estimate over the entire workspace. This can be used to compute optimal motion plans which take into account not only the distance to the goal, but also the number of times the robot's position must be estimated. This number is determined by the accuracy of each estimate (from the value we have computed at the position the measurement was made) and how accurately does the robot need to know its location (less accurate when it is far from an obstacle, and more accurate when it is near an obstacle or close to a goal position that has to be reached at a certain precision). This problem is discussed in detail in [12] and [13].

The inverse problem is also interesting. In this case, requirements are made on the accuracy of the robot's position at certain places and on certain paths in the workspace. The task is to determine where to place the landmarks in the workspace. This problem has been investigated in [14]. Our method can be used as the basic building block in research on these types of problems.

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Characterizations of Localization Accuracy of Fixtures

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***Abstract*—In this paper, an analysis is presented of the problem of characterizing the accuracy of deterministic localization of fixtures. In a statistical framework, the positioning accuracy of the workpiece localized by the locators of a fixture is described by a symmetric, positive definite *accurateness* matrix (or *variance* matrix). The accurateness (variance) matrix is identified as having similar structural properties to the stiffness (compliance) matrix of an unloaded, stable robot grasp. This connection leads us to describe a set of frame-invariant characteristic parameters with geometric interpretation. The principal translational accuratenesses and rotational variances are defined for constructions of frame-invariant quality measures for a meaningful comparison of different locating schemes. Examples are presented to illustrate the concept and usefulness of the characterizing properties in optimizing a fixture layout.**

***Index Terms*—Accurateness, fixture localization, fixturing, grasping, stiffness.**

I. Introduction

Proper fixture design is crucial to product quality in terms of precision and accuracy in part fabrication and assembly. Fixtures, usually consisting of clamps and locators, must be capable of positioning, holding, and supporting a workpiece during machining, assembly, or inspection. Among the three major functions, positioning or localization is the primary function, and the positioning accuracy is an essential measure of the performance quality of a fixture. This paper focuses on the aspect of localization accuracy and describes various characterizations with frame-invariant quality measures. The measures may be used to quantify or evaluate different locating schemes in optimal design of fixtures.

Literature on general fixturing techniques is substantial [1]. There are several formal methods for analyzing performance of a fixture based on the popular screw theory, dealing with issues such as kinematic closure [2], [3], contact types, and friction effects [4]. The focus is usually on the century-old concept of *form (force) closure* [5], which has been extensively studied in the field of robotics in recent years [6]–[8]. The problem of designing modular fixtures has gained extensive attention recently [9].

Prior work on quantifying effectiveness of a robotic grasp or a fixture has mostly focused on the *state of equilibrium*. Equilibrium is involved in the total restraint (or closure) of the object or workpiece. There are two major ways to quantify an equilibrium grasp or fixture. In considering the rigid-body mechanics only, measures of the contact constraints provided by robot figures or fixture contacts may be expressed in terms of the contact force such as the maximum of the normalized equilibrium forces [7], [10], [11]. Such measures are usually dependent on the choice of coordinate frames—an optimal grasp or fixturing scheme under one choice of reference frame may become nonoptimal under another. Another approach is to consider the stiffness matrix of a grasp or fixture with an elastic contact model [6], [12]–[14]. In recent years, it has been realized that the stiffness matrix of an *unloaded equilibrium* has a special structure defining the elastic couplings [14]–[17]. A proper characterization of the stiffness matrix has been shown to have strong implications in practice [17].

The functional requirement considered in this paper is the workpiece localization, which is different from the concept of equilibrium. Localization establishes a desired spatial relationship of the workpiece with respect to a fixture reference frame with *unique* and *accurate* location (both position and rotation). For a fixture, locators are used primarily for this function to eliminate the six degrees of freedom of the workpiece, which is often referred to as *deterministic localization* [2]. But, as passive elements, the locators in deterministic localization do not necessarily form an equilibrium [18]. Once positioning of the workpiece is accomplished by the locators, a clamping device must be applied to provide a complete restraint (or force closure) of the workpiece against any external forces on the workpiece. Only then is a stable equilibrium established. Our focus here is on the accuracy of the *deterministic localization*.

The localization accuracy is subject to positioning variability of the locators and geometric variability of the workpiece. Its proper qualification is an important issue in fixture design, as it may be considered as a major criterion for the designer to use in choosing from a large number of feasible locating schemes. The prior research on this issue is scarce, with some case-specific and limited measures available [19], [20]. This paper builds on a statistical framework of fixture analysis and presents *frame-invariant* characterizations of the localization accuracy. First, a general representation of localization accuracy is described by an *accurateness matrix*. Then, we recognize some key similarities of this matrix with the stiffness matrix of an unloaded stable equilibrium in grasping or fixturing. This realization

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